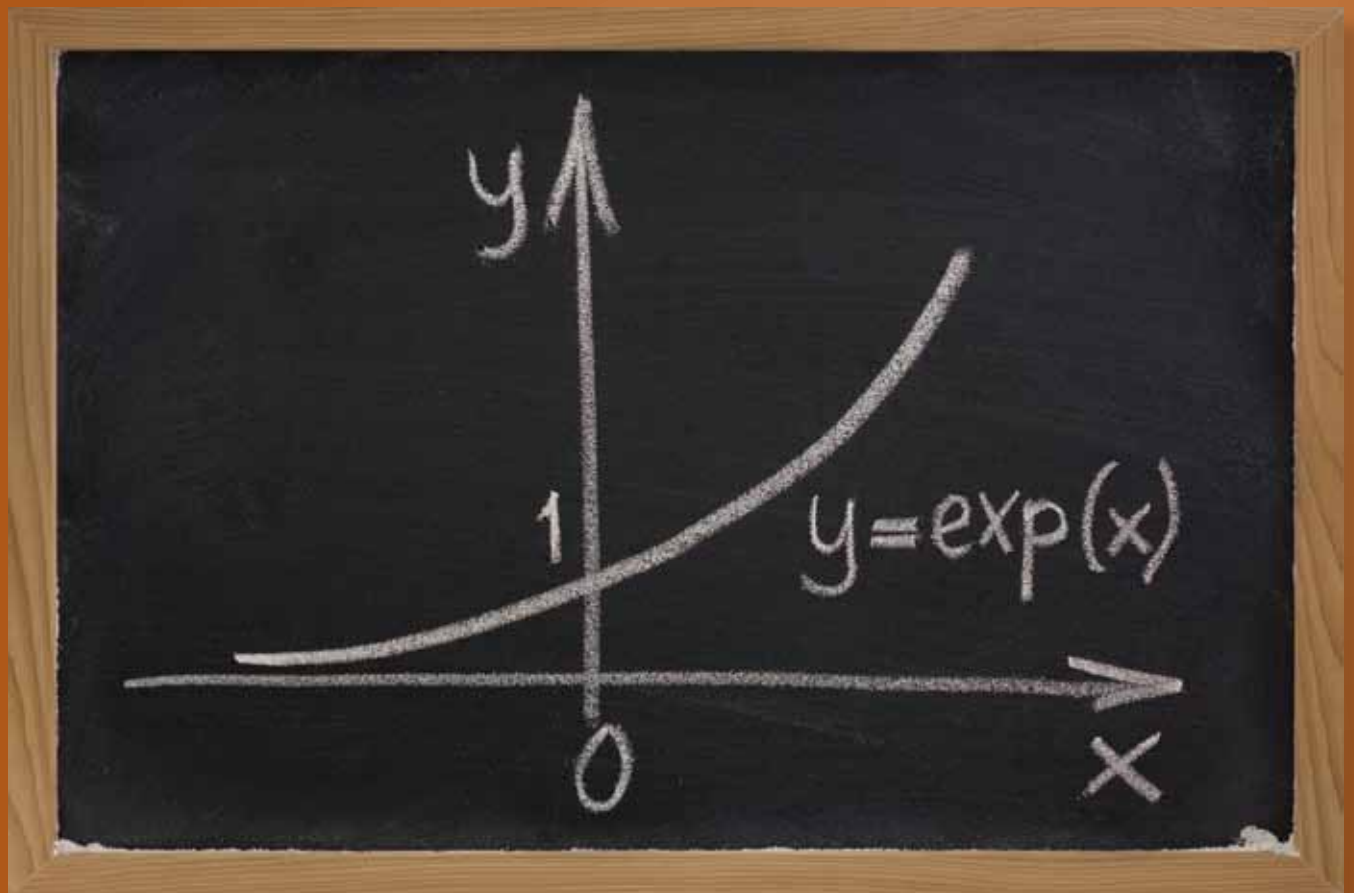THE GREAT COURSES®

Algebra II

Course Workbook

Professor James A. Sellers
The Pennsylvania State University

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Corporate Headquarters
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Phone: 1-800-842-2412
Fax: 703-378-3819
www.thegreatcourses.com

James A. Sellers, Ph.D.

Professor of Mathematics and Director of Undergraduate Mathematics
The Pennsylvania State University

Professor James A. Sellers is Professor of Mathematics and the Director of Undergraduate Mathematics at The Pennsylvania State University. He has served as the Director of Undergraduate Mathematics at Penn State since 2001. From 1992 to 2001, he was a mathematics professor at Cedarville University in Ohio.

Professor Sellers received his B.S. in Mathematics in 1987 from The University of Texas at San Antonio, the city in which he was raised. He then went to Penn State University, where he received his Ph.D. in Mathematics in 1992. There he worked under the direction of his Ph.D. adviser, David Bressoud, and learned a great deal about the beauty of number theory, especially the theory of integer partitions. Professor Sellers has written numerous research articles in the area of partitions and related topics; to date, at least 60 of his papers have appeared in a wide variety of peer-reviewed journals. He is especially fond of coauthoring papers with his undergraduate students; his list of coauthors includes 8 undergraduates he has mentored during his career. Professor Sellers was privileged to spend the spring semester of 2008 as a visiting scholar at the Isaac Newton Institute at the University of Cambridge, pursuing further studies linking the subjects of partitions and graph theory.

Professor Sellers's teaching reputation is outstanding. He was named the Cedarville University Faculty Scholar of the Year in 1999, a truly distinct honor at the institution. At Penn State, he received the Mary Lister McCammon Award for Distinguished Undergraduate Teaching from his department in 2005. One year later, he received the Mathematical Association of America Allegheny Mountain Section Award for Distinguished Teaching. Since then, he has also received the Teresa Cohen Mathematics Service Award from the Penn State Department of Mathematics (in 2007) and the Mathematical Association of America Allegheny Mountain Section Mentoring Award (in 2009).

Professor Sellers has enjoyed many interactions at the high school and middle school levels. He served as an instructor of middle school students in the TexPREP program in San Antonio, Texas, for 3 summers. He worked with Saxon Publishers on revisions to a number of their high school–level textbooks in the 1990s. As a home educator and father of 5, he has spoken to various home education organizations about mathematics curricula and teaching issues.

Professor Sellers is well-known as an entertaining and gifted speaker. He has spoken at numerous college, university, and high school venues about partitions and combinatorics. He has also spoken at a number of conferences and seminars across the United States, sharing results related to his own research as well as his views on teaching and advising undergraduate students. ∎

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Algebra II

Scope:

T his course starts with a review of concepts from Algebra I. You'll learn how to sharpen your problem-solving skills in linear and quadratic equations and then work your way up to conic sections, roots and radicals, exponential and logarithmic functions, and elementary probability.

As you gain confidence by working through problems with Professor Sellers, you will see that the ideas behind algebra are wonderfully interconnected, that there are often several routes to a solution, and that the concepts and procedures discussed have a host of applications. For example, the function is one of the simplest and most powerful ideas in algebra. You will see how plotting a function as a graph often brings its key properties sharply into focus.

In the second half of the course, you will progress to performing complex operations on polynomials. You'll become adept at working with conic sections, roots and radicals, and exponential and logarithmic functions. As with so many tools in algebra, these concepts are simple, but their applications are powerful.

With your growing mathematical maturity, you will learn to deploy an arsenal of formulas, theorems, and guidelines that provide a deeper understanding of patterns in algebra while allowing you to analyze and solve equations quickly. These useful techniques include the vertical line test, the quadratic formula, and the fundamental theorem of algebra.

The course culminates with lectures on combinatorics and probability, which have applications that range from the scientific to the entertaining—such as how many different 3-topping pizzas can be made from a selection of 8 toppings! ∎

An Introduction to Algebra II
Lesson 1

Topics in This Lesson

- Course overview.

- The order of operations.

- The laws of exponents.

Summary

Welcome to *Algebra II*! We will cover a wide variety of topics in this course. Two of the most important topics that we must keep in mind throughout this course are order of operations and the laws of exponents.

"Order of operations" refers to the order in which we must perform the usual operations of addition, subtraction, multiplication, division, and exponentiation. Here are the basic ideas to remember:

1. Simplify whatever is inside parentheses first.

2. Reading left-to-right, perform all exponentiations.

3. Reading left-to-right, perform all multiplications and divisions.

4. Reading left-to-right, perform all additions and subtractions.

Next, we need to think about the laws of exponents, which are listed below.

1. $x^a \cdot x^b = x^{a+b}$

2. $\dfrac{x^a}{x^b} = x^{a-b}$

3. $(x^a)^b = x^{ab}$

Here x usually refers to a positive real number while a and b can be any real numbers.

Definitions and Formulas

laws of exponents:

1. $x^a \cdot x^b = x^{a+b}$

2. $\dfrac{x^a}{x^b} = x^{a-b}$

3. $(x^a)^b = x^{ab}$

Examples

Example 1

Using the order of operations, simplify the following:

$$10 - 5 \times (8 - 2) + 17 - 13 \times 2.$$

First, we simplify the $8 - 2$ in the parentheses to get 6. That gives us

$$10 - 5 \times 6 + 17 - 13 \times 2.$$

Next, we perform the multiplications (from left to right):

$$\begin{aligned} &10 - 30 + 17 - 13 \times 2 \\ =\ &10 - 30 + 17 - 26. \end{aligned}$$

Lastly, going left-to-right, perform the additions and subtractions:

$$\begin{aligned} &10 - 30 + 17 - 26 \\ =\ &-20 + 17 - 26 \\ =\ &-3 - 26 \\ =\ &-29. \end{aligned}$$

Example 2

Compute $5 + 3 \times 4^2$.

The first thing to do is the exponentiation:

$$\begin{aligned} &5 + 3 \cdot 4^2 \\ =\ &5 + 3 \cdot 16. \end{aligned}$$

Next, we must do the multiplication:

$$= 5 + 3 \cdot 16$$
$$= 5 + (3 \cdot 16)$$
$$= 5 + 48.$$

Then we do the addition:

$$5 + 48 = 53.$$

Example 3

Using the laws of exponents, simplify the following.

a. $2^5 \cdot 2^3$

b. $\dfrac{4^9}{4^3}$

c. $\dfrac{3^2}{3^8}$

d. $(2^7)^3$

e. $((-2)^3)^4$

Solutions

a. $2^5 \cdot 2^3 = 2^{(5+3)} = 2^8$

b. $\dfrac{4^9}{4^3} = 4^{(9-3)} = 4^6$

c. $\dfrac{3^2}{3^8} = 3^{(2-8)} = 3^{-6}$ or $\dfrac{1}{3^6}$

d. $(2^7)^3 = 2^{7 \cdot 3} = 2^{21}$

e. $((-2)^3)^4 = (-2)^{3 \cdot 4} = (-2)^{12} = 2^{12}$ since 12 is even

Example 4

Compute the following using order of operations and laws of exponents.

$$((5-3)^2)^3 + 8 \times 4$$

We must begin by simplifying what is inside the parentheses, then using the laws of exponents, and finally working through the order of operations (remembering that multiplication takes precedence over addition).

$$
\begin{aligned}
&((5-3)^2)^3 + 8 \times 4 \\
&= (2^2)^3 + 8 \times 4 \\
&= 2^6 + 8 \times 4 \\
&= 64 + 8 \times 4 \\
&= 64 + 32 \\
&= 96
\end{aligned}
$$

Common Errors

- Forgetting to simplify whatever is inside a set of parentheses before beginning the exponentiation, multiplication, division, addition, and subtraction.

- Not memorizing the laws of exponents.

Study Tips

- If you are having trouble remembering the laws of exponents, make a small set of flashcards for memorization.

- Practice, practice, practice! Doing lots of examples of these kinds of problems will be really helpful as we move forward.

Problems

Simplify the following using the order of operations.

1. $14 \times (8-5) + 2^5$

2. $(-5-3)^2 + 100$

3. $17 + 3 \cdot 5 - 4 \div 2 + 19$

4. $5 + 3^2 - 6 \cdot 7$

5. $1 + 2 - 3 \cdot 4 \div (5 - 2)$

Simplify the following using the laws of exponents.

6. $(3^2)^5$

7. $((-2)^3)^3$

8. $4^7 \div 4^2$

9. $5^2 \div 5^{-4}$

10. $2^3 \cdot 2^{-6} \cdot 2^8$

Solving Linear Equations
Lesson 2

Topics in This Lesson

- Understanding the idea of a solution of a linear equation.

- Learning how to solve a wide variety of linear equations.

Summary

Our goal in this lesson is to begin to learn how to solve linear equations.

Definitions and Formulas

identity: An equation that is true for every possible value of the variable.

solution of a linear equation: A value of the variable that makes the equation true.

Examples

Example 1

Solve $5x + 4 = -11$.

We need to get the variable x by itself. The first step is to move the 4 to the other side of the equation. (You might ask—why the 4 and not the 5? Believe it or not, the answer is wrapped up in the order of operations! To isolate x, we must undo the multiplication by 5 and the addition of 4 *in reverse order*. Hence, we start by getting rid of the 4.) To do so, we subtract 4 from both sides.

$$\begin{array}{r} 5x + 4 = -11 \\ \underline{-4 \quad -4} \\ 5x = -15 \end{array}$$

Next, we need to get the 5 away from the x. This is done by dividing by 5 or multiplying by 1/5.

$$\begin{array}{c} \dfrac{5x}{5} = \dfrac{-15}{5} \\ x = -3 \end{array}$$

We should check to make sure this is indeed a solution of the original equation.

$$5x + 4 = -11$$
$$5(-3) + 4 = -11$$
$$-15 + 4 = -11$$
$$-11 = -11$$

This is true, so it is a correct solution.

Example 2

Solve $-x + 3 = 5x + 63$.

Our ultimate goal is to get all the variables on one side of the equation and the constants on the other side. Let's start by getting the constants over to the right-hand side. Well, the 63 is already over there! So we just need to get rid of the 3 on the left-hand side by subtraction.

$$\begin{array}{r} -x + 3 = 5x + 63 \\ -3 \qquad -3 \\ \hline -x = 5x + 60 \end{array}$$

Next, we want to get the x's to the left-hand side. The −x is already there, so it's just the 5x that we must move. How? Subtract $5x$ from both sides.

$$\begin{array}{r} -x = 5x + 60 \\ -5x \quad -5x \\ \hline -6x = 60 \end{array}$$

Remember, $-x$ is the same as $-1x$. So $-x - 5x$ is the same as $-1x - 5x$, which is $-6x$.

Now we just need to get that −6 out of the way. How? Divide by −6 on both sides.

$$\frac{-6x}{-6} = \frac{60}{-6}$$
$$x = -10$$

So our solution is $x = -10$, if we've done everything correctly. Let's check:

$$-x + 3 = 5x + 63$$
$$-(-10) + 3 = 5(-10) + 63$$
$$10 + 3 = -50 + 63$$
$$13 = 13$$

This is true, so our solution is correct.

Example 3

Solve $3(8x - 6) = 21x + 9$.

Let's start by distributing the 3 on the left-hand side.

$$3 \times 8x - 3 \times 6 = 21x + 9$$
$$24x - 18 = 21x + 9$$

Now, let's put the x's on the left-hand side.

$$24x - 18 = 21x + 9$$
$$\underline{-21x \qquad\quad -21x}$$
$$3x - 18 = 9$$

Now we add 18 to both sides of this new equation.

$$3x - 18 + 18 = 9 + 18$$
$$3x = 27$$

Now just divide by 3 to isolate the x.

$$x = 9$$

So our solution is $x = 9$. We should check this to make sure it is correct.

$$3(8x - 6) = 21x + 9$$
$$3(8(9) - 6) = 21(9) + 9$$
$$3(72 - 6) = 189 + 9$$
$$3(66) = 189 + 9$$
$$198 = 198$$

Yes! The value $x = 9$ really is a solution of the original equation.

Common Errors

- You may believe you have made a mistake when you arrive at what appears to be an error, like $2 = 4$. Assuming you have not made any errors in your arithmetic or algebra, this is simply an indication that the original equation had no solutions!

- Forgetting to distribute appropriately. When we encounter $3(8x - 6)$, we may be tempted to write this as $3 \times 8x - 6$, or $24x - 6$. But this is incorrect; the 3 must be distributed to the $8x$ as well as to the -6.

Study Tips

- Take your time when working through the steps to solve a linear equation. There can often be a number of different steps to complete, and one little mistake on any one of these steps can mess up the whole problem.

- Write each step clearly and neatly; don't try to do a bunch of work in your head!

Problems

Solve each of the following equations.

1. $x + 5 = -5x + 5$

2. $x - 1 = 6x + 2x - 8$

3. $5x - 14 = 8x + 4$

4. $-4(-6x - 3) = 12$

5. $-(7 - 4x) = 9$

6. $-18 - 6x = 6(1 + 3x)$

7. $2(4x - 3) - 8 = 2x + 4$

8. $3x - 5 = -8(5x + 6)$

9. $-3(4x + 3) + 4(6x + 1) = 19$

10. $28x - 22 = -4(-7x + 1)$

Solving Equations Involving Absolute Values
Lesson 3

Topics in This Lesson

- Understanding the idea of absolute values (in relation to the real number line).

- Learning how to solve equations that involve absolute values.

Summary

In this lesson, we continue to discuss solving equations, including those that contain absolute values as terms. This means we need to understand what absolute values are and how to simplify terms that involve absolute values. Equations that contain absolute values can often be rewritten so that the absolute value symbol no longer appears. Once this is done, the equations look like equations we have solved in the past.

Definitions and Formulas

absolute value: The absolute value of a number x is typically denoted $|x|$ and is the distance from x to the origin on the number line.

extraneous solution: A solution of a simplified version of an equation that is not a solution of the original equation.

origin: The location where zero is placed on a real number line.

Examples

Example 1

Solve $|2x - 4| = 14$.

When solving such absolute value problems, we must convert them into 2 separate equations: $2x - 4 = 14$ and $2x - 4 = -14$. This is because the quantity on the left-hand side of the original equation could be 14 steps to the *left* of the origin or 14 steps to the *right* of the origin on the number line. So solving $|2x - 4| = 14$ is really solving 2 different problems.

$$2x - 4 = 14$$
$$2x = 18 \quad \text{(by adding 4 to both sides)}$$
$$x = 9 \quad \text{(by dividing both sides by 2)}$$

So it appears that $x = 9$ is a solution. Let's check by plugging 9 back in for x in the original problem.

$$|2(9) - 4| = |18 - 4| = |14| = 14$$

Correct. OK, now let's do the other half of the problem.

$$2x - 4 = 14$$
$$2x = -10 \text{ (by adding 4 to both sides)}$$
$$x = -5 \text{ (by dividing both sides by 2)}$$

So it looks like $x = -5$ is a second solution. Let's check:

$$|2(-5) - 4| = |-10 - 4| = |-14| = 14. \text{ Correct.}$$

It turns out that our original equation actually has 2 solutions, $x = 9$ and $x = -5$.

Example 2

Solve $|3x + 5| = 5x + 2$.

Again, we split this into 2 equations and solve them separately.

$$3x + 5 = 5x + 2 \text{ or } 3x + 5 = -(5x + 2)$$

$$3x + 5 = 5x + 2$$
$$5 = 2x + 2$$
$$3 = 2x$$
$$x = 3/2$$

$$3x + 5 = -(5x + 2)$$
$$3x + 5 = -5x - 2$$
$$8x + 5 = -2$$
$$8x = -7$$
$$x = -7/8$$

Let's check to see if these 2 numbers are both solutions. First we look at 3/2.

$$|3x + 5| = 5x + 2$$
$$|3(3/2) + 5| = 5(3/2) + 2$$
$$|9/2 + 5| = 15/2 + 2$$
$$|9/2 + 10/2| = 15/2 + 4/2$$
$$|19/2| = 19/2$$
$$19/2 = 19/2$$

So $x = 3/2$ really is a solution of the original equation. What about $-7/8$?

$$| 3x + 5 | = 5x + 2$$
$$| 3(-7/8) + 5 | = 5(-7/8) + 2$$
$$| -21/8 + 5 | = -35/8 + 2$$
$$| -21/8 + 40/8 | = -35/8 + 16/8$$
$$| 19/8 | = -19/8$$
$$19/8 = -19/8$$

This is not true, so $x = -7/8$ is not a solution. It is called an extraneous solution. The only true solution of the original equation is $x = 3/2$.

Example 3

Solve $4 | x + 8 | = 56$.

We begin by dividing both sides by 4.

$$| x + 8 | = 14$$

We split this into 2 equations: $x + 8 = 14$ and $x + 8 = -14$. Now we solve each separately.

$$x + 8 = 14$$
$$x = 6$$

$$x + 8 = -14$$
$$x = -22$$

So the 2 potential solutions are $x = 6$ and $x = -22$. Let's check by plugging these into the original equation, beginning with 6.

$$4 | x + 8 | = 56$$
$$4 | 6 + 8 | = 56$$
$$4 | 14 | = 56$$
$$4 \cdot 14 = 56$$
$$56 = 56$$

Correct. So $x = 6$ is truly a solution. Now let's plug in -22.

$$4\,|\,x + 8\,| = 56$$
$$4\,|\,-22 + 8\,| = 56$$
$$4\,|\,-14\,| = 56$$
$$4 \cdot 14 = 56$$
$$56 = 56$$

Correct. So $x = -22$ is also a solution.

Common Errors

- Converting an absolute value equation by simply dropping the absolute value symbols. For instance, turning $|\,3x + 1\,| = 19$ into $3x + 1 = 19$. If you don't also consider the equation $3x + 1 = -19$, then you will likely miss a solution of the original equation.

- Forgetting to check your solutions. All solutions should be checked! Some of the solutions are extraneous, which means they are not solutions at all!

Study Tips

- Check all solutions carefully to make sure they are indeed solutions.

- Remember that absolute values can never be negative. So an equation such as $|\,x\,| = -5$ can never have a solution.

Problems

Find the solutions of the following problems.

1. $|\,-x + 7\,| = 19$

2. $|\,2x + 3\,| = 7$

3. $-2|\,x + 2\,| + 12 = 0$

4. $|\,3x + 2\,| + 8 = 0$

5. $|\,2x + 12\,| = 7x - 3$

6. $|x + 6| = 3x$

7. $3|x + 7| = 36$

8. $|(1/2)x + 2| = 8$

9. $|2x + 10| = 0$

10. $|2x + 7| = x - 4$

Linear Equations and Functions
Lesson 4

Topics in This Lesson

- Linear equations and functions, especially as they relate to their graphs.

- The slope-intercept formula and point-slope formula for lines, and how each of these formulas is useful in its own right.

- The concept of parallel lines and how to identify them.

- The concept of perpendicular lines and how to identify them.

Summary

The bulk of this lesson is spent on 3 main subjects: the slope-intercept form of a line, the point-slope form of a line, and the special pairs of lines called parallel and perpendicular lines. It is important to understand how lines are represented using either the slope-intercept form or the point-slope form of a line. These 2 different representations tell us something different about a line, and they also require slightly different information in order to use them.

Definitions and Formulas

parallel: Two nonvertical lines with the same slope and different y-intercepts are said to be parallel to one another.

perpendicular lines: Two lines that intersect at a right angle. If neither of the 2 lines is vertical, then we can say that the 2 lines are perpendicular if the slopes of the 2 lines are negative reciprocals of one another, or if the product of the 2 slopes equals -1.
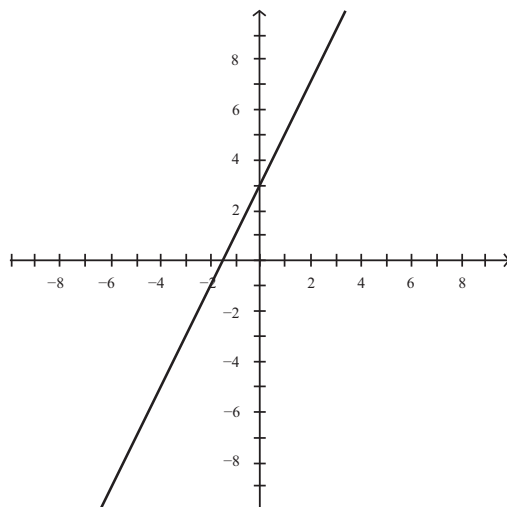
point-slope form: The equation of a line that passes through the point (x_1, y_1) and has slope m is $y - y_1 = m(x - x_1)$.

slope-intercept form: An equation of a line $y = mx + b$, where m is the slope of the line and b is the y-intercept of the line.

Examples

Example 1

Determine the slope-intercept form of the equation of this line.

In order to write the slope-intercept form of this line, we need to determine the slope and the y-intercept. The y-intercept here is easy to see; it is $b = 3$. Now we need the slope. We can choose any 2 points to determine the slope. Let's use $(0, 3)$ and $(1, 5)$.

$$\text{Slope} = m = \frac{5-3}{1-0} = \frac{2}{1} = 2$$

Thus, the equation of this line is $y = mx + b$, or $y = 2x + 3$.

Example 2

Write the equation of the line that has slope 5 that passes through the point $(-1, -3)$.

With the slope-intercept form of the equation, we would have some difficulty determining the equation of the line. But this is exactly the information we need to write the equation of the line in point-slope form.

$$y - (-3) = 5(x - (-1)), \text{ or } y + 3 = 5(x + 1)$$

Example 3

Determine whether the lines given by

$$y = -2x + 7 \text{ and } 10x + 5y = 12$$

are parallel.

To solve this problem, we need to determine the slope of each line. It's clear that the line represented by the first equation has slope −2. What about the slope of the second line? We must perform a bit of algebra to rewrite that line in slope-intercept form:

$$
\begin{aligned}
10x + 5y &= 12 \\
5y &= -10x + 12 \\
(1/5)5y &= (1/5)(-10x + 12) \\
y &= (1/5)(-10x) + (1/5)(12) \\
y &= -2x + 12/5
\end{aligned}
$$

Note that this line also has slope −2. And the 2 lines have different *y*-intercepts (7 and 12/5). Therefore, they are indeed parallel.

Example 4

Find an equation of the line that is parallel to the line given by $y = 5x - 1$ and has *y*-intercept (0, 4).

Notice that the slope of the line that was given to us is 5, so the line we want must also have slope 5. Also, the *y*-intercept was given to us as 4. So we can use the slope-intercept form to quickly write down the line we want: $y = 5x + 4$.
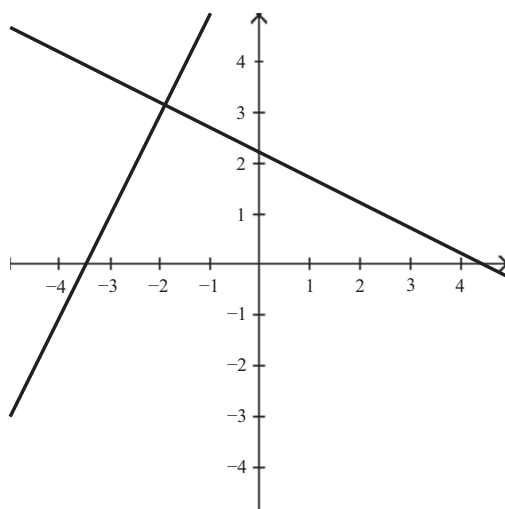
Example 5

Determine whether the lines given by $y = 2x + 7$ and $2x + 4y = 9$ are perpendicular.

To determine whether these are perpendicular, we need to compare their slopes. The slope of the first line is easy; it is 2. Now we need to find the slope of the second line. Let's rewrite it in slope-intercept form.

$$
\begin{aligned}
2x + 4y &= 9 \\
4y &= -2x + 9 \\
y &= (-1/2)x + (9/2)
\end{aligned}
$$

So the slope of the second line is −1/2. Since 2 and −1/2 are negative reciprocals of one another, we see that the 2 lines we were given are indeed perpendicular. We can also visualize this by sketching the 2 graphs and noting their perpendicularity.

Common Errors

- When using the point-slope form of the equation of a line, it is easy to make mistakes with the arithmetic, especially when dealing with negative numbers. For example, in the problem above where the line passed through the point $(−1, −3)$, the equation was $y − (−3) = 5(x − (−1))$, or $y + 3 = 5(x + 1)$. It would have been very tempting to write that equation as $y − 3 = 5(x − 1)$, but this would have been incorrect.

- Also when using the point-slope form of the equation of a line, it is very tempting to switch the roles of the x-coordinate and the y-coordinate of the point through which the line travels. For example, if the line is to have slope 1 and pass through the point $(2, 3)$, then the point-slope form of the line is $y − 3 = 1(x − 2)$. Make sure you don't fall into the trap of incorrectly writing the equation as $y − 2 = 1(x − 3)$, switching the 2 and 3.

Study Tips

- For all problems in this course that require you to draw a graph, it would be wise to use graph paper or grid paper. This allows for much more accurate drawing and will probably help avoid errors in sketching the graphs.

- Remember that lines with negative slope are drawn from "high to low" as we read them from left to right, while lines with positive slope are drawn from "low to high" as we read them from left to right.

- Remember also that lines with very large slope are closer to vertical, while lines with very small slope (near zero) are closer to horizontal.

Problems

Find the equation of the line satisfying the following requirements.

1. The line must have slope 3 and pass through the point (4, 2).

2. The line must be parallel to the line $y = 7x + 3$ and have y-intercept 4.

3. The line must be perpendicular to the line $y = x/2 + 5$ and pass through the origin.

4. The line must pass through the points (1, 2) and (4, 8).

Determine whether the following pairs of lines are parallel to one another.

5. $x + y = 9$; $x - y = 2$

6. $y = 4x + 3$; $8x - 2y = 24$

7. $4x + 2y = 10$; $10x + 5y = 25$

Determine whether the following pairs of lines are perpendicular to one another.

8. $y = 6x + 9$; $x + 6y = 14$

9. $y = 3x - 2$; $y = x/3$

10. $y = 7$; $x = -2$

Graphing Essentials
Lesson 5

Topics in This Lesson

- Sketching the graph of a line, given its equation.

- Learning different ways to transform the graph of one function into the graph of another function by shifting the graph to the left or right, or up or down, and possibly even flipping the graph around one of the axes.

Summary

The focus of this lesson is very visual—learning to sketch the graphs of linear functions (or lines) as well as to construct the graphs of relatively complicated functions in a step-by-step fashion, using shifts and flips of the graphs of more basic functions.

Examples

Example 1

Sketch the graph of the equation $y - 6 = \frac{1}{2}(x - 3)$.

The key here is to recognize that this is the equation of a line, and the equation is in point-slope form. So we just need to interpret what we have been given. The slope of the line is 1/2. Also, we know the coordinates of a point through which the line passes. That point is (3, 6). Now we can sketch our graph. We know it goes through (3, 6), and we know that the "rise over run" is 1/2. So for every 1 unit we go up, we need to go over 2 to the right. So another point on the line is (5, 7). − 3 + 2 = 5, and 6 + 1 = 7. Now plot those 2 points and connect the dots.
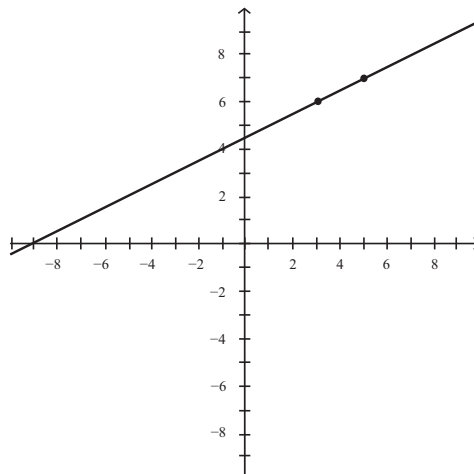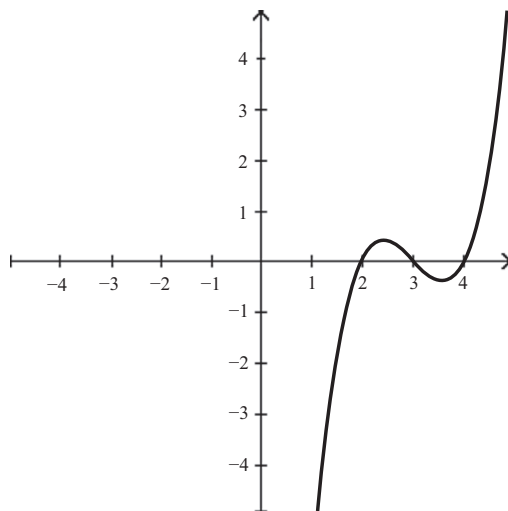
Example 2

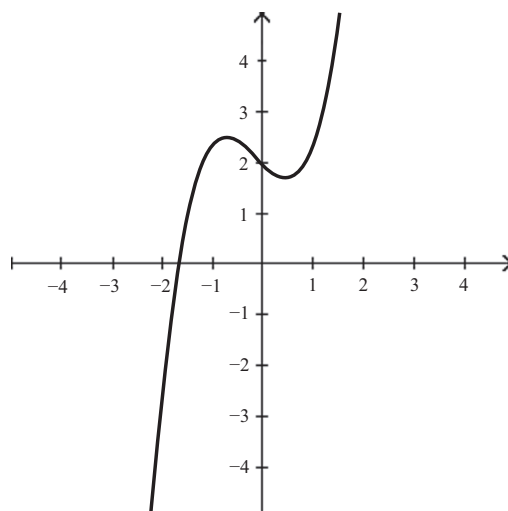Starting with the graph of $y = x^3 - x$, determine the graphs of $y = (x-3)^3 - (x-3)$ and $y = x^3 - x + 2$.
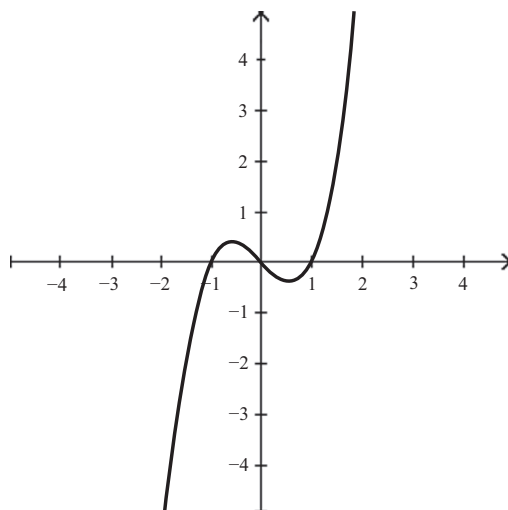
We begin with the graph of $y = x^3 - x$.

We will talk much more about where this comes from in future lessons. In order to get the graph of $y = (x-3)^3 - (x-3)$, we can move the graph of $y = x^3 - x$ over 3 units to the right.

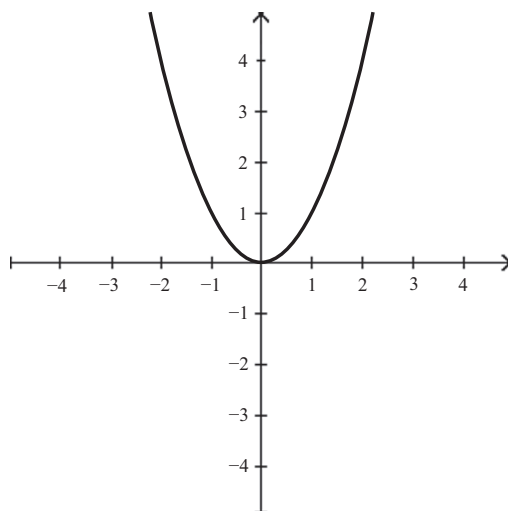In order to get the graph of $y = x^3 - x + 2$, we simply move the graph of $y = x^3 - x$ up 2 units.

Example 3

Given the graph of $y = x^2$, draw the graph of $y = (x+3)^2 - 4$.

We begin with the graph of $y = x^2$.

Now there are 2 steps we must take. First of all, notice that the x has been changed to $x + 3$. This means that the new graph needs to shift to the left by 3 units. Also, we see that the whole function has had −4 added to it. That means the graph needs to shift down by 4 units. So we take the graph of $y = x^2$ and shift it to the left by 3 units and down by 4 units. This gives us the new graph below.
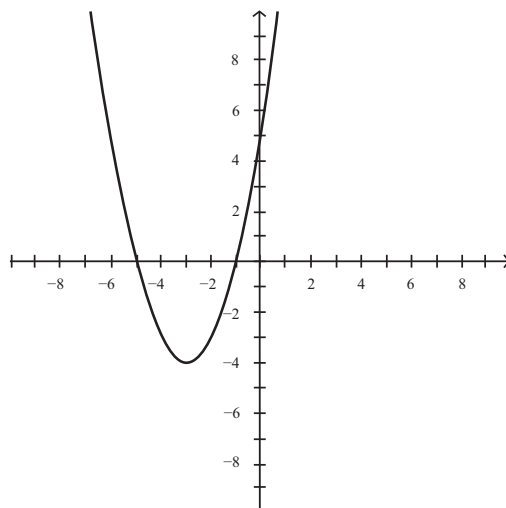
Common Errors

- It is very common to mix up the rules for when to shift a graph left or right versus up or down. In particular, some students believe that the graph of $y = (x - 3)^2$ is the graph of $y = x^2$ shifted to the left by 3 units. This is a somewhat natural mistake to make, since we typically think of negative numbers as being to the left of the origin. But this is not correct in graphing. The graph of $y = (x - 3)^2$ is the graph of $y = x^2$ shifted to the right by 3 units.

Study Tips

- When sketching the graphs of functions, use graph paper or grid paper.

- When you are not sure how to start a problem, remember that you can always plot a few points to try to see the shape of the graph.

- Similarly, if you just want to sketch the graph of a line, you just have to plot 2 points on the line and connect the dots.

Problems

Sketch the graphs of the following linear functions.
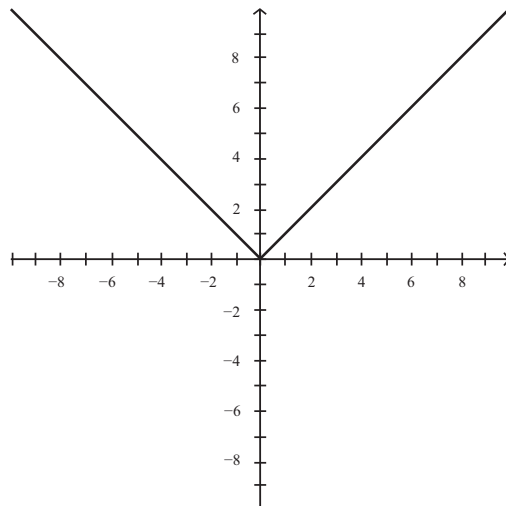
1. $y = -2x + 3$

2. $y = -3$

3. $3x + y = 4$

4. $6x - 3y = 18$

5. $2(3x + y) = 14$

The graph of $y = |x|$ is below.

Determine the graph of each of the following.

6. $y = |x| + 2$

7. $y = |x - 3|$

8. $y = |x + 1| + 4$

9. $y = -|x|$

10. $y = |2x|$

Functions—Introduction, Examples, Terminology
Lesson 6

Topics in This Lesson

- The concept of a function.

- The meaning of the domain and range of a function, and how to identify them.

- The vertical line test, which helps us identify from its graph whether an equation is a function.

Summary

In this lesson, we study the very important concept of functions. We consider several equations, some of which are functions and others of which are not. We learn to determine the domain and range of a function and how to look at the graph of an equation to see if that graph represents a function (thanks to the vertical line test).

Definitions and Formulas

composition of 2 functions: For functions $f(x)$ and $g(x)$, this is the new function obtained by inserting $g(x)$ into the function $f(x)$. The new function is often denoted $f(g(x))$.

domain of a function: The set of input values of a function.

function: A set of pairs of input values and output values in which each input value is assigned to exactly 1 output value.

range of the function: The set of output values of a function.

vertical line test: A test used to determine whether a graph represents a function. If you can draw a vertical line that crosses through at least 2 points on a graph, then the graph does not represent a function.

Examples

Example 1

Determine the domain and range of the function $f(x) = \sqrt{x-3}$.

First we should think about the domain. Can we ever compute something like $\sqrt{-9}$? No. Notice that it's *not* −3 (although you might be tempted to say so). It's not −3 because $(-3)(-3) = 9$, not −9. In fact, $\sqrt{x-3}$ can only be computed if $x - 3 \geq 0$. That's the same as $x \geq 3$. So the domain of this function is $[3, \infty)$. What about the range? As we have already hinted, the output value of any square root that you take is going to be zero or positive. $\sqrt{0} = 0, \sqrt{1} = 1, \sqrt{4} = 2$, and so on. So what's the range? It is the interval $[0, \infty)$.

Example 2

Let $f(x) = 2x + 4$ and $g(x) = 3x + 2$. Find the composition of f and g as well as the composition of g and f.

The composition of f and g is denoted $f(g(x))$. Notice how it looks like we plugged g right into f. The notation is meant to give you that impression because that's exactly what we do. We take the rule for $g(x)$, which is the quantity $3x + 2$, and we plug that in for x in the rule for $f(x)$. So we then have

$$f(g(x)) = 2(3x + 2) + 4,$$

which we can simplify to
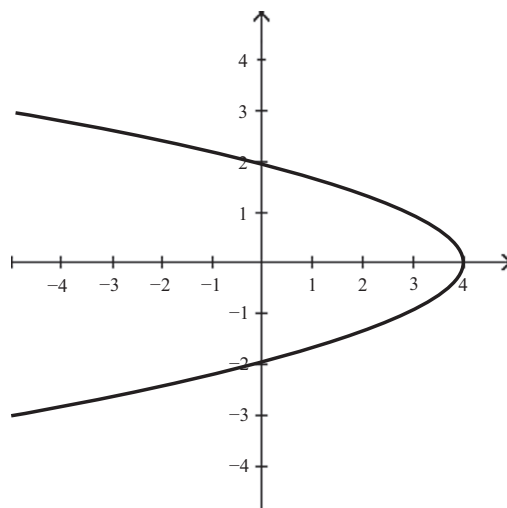
$$f(g(x)) = 2(3x + 2) + 4 = 6x + 4 + 4 = 6x + 8.$$

Here's something cool: You can also find $g(f(x))$.

$$g(f(x)) = 3(2x + 4) + 2 = 6x + 12 + 2 = 6x + 14.$$

Notice how $f(g(x)) \neq g(f(x))$? Most of the time, these 2 will not be equal to one another. Sometimes they are equal, and that will signal that f and g have a special relationship with one another.

Example 3

Is this the graph of a function?

No. I can draw a vertical line that crosses this graph at 2 points (look at the *y*-axis itself—it is crossing 2 points of the graph!). So this does not represent a function.

Common Errors

- Thinking that the square root of a negative number exists as a real number, for example, that $\sqrt{-16} = -4$. But this is not true, because $(-4)(-4) = 16$, not -16. It is important to remember that the square root of a negative number is not a real number.

Study Tips

- When trying to determine the domain of a function, it is often useful to look for the numbers that are not in the domain and exclude them. The most common reasons a number might need to be excluded from the domain of a function are if that number causes division by zero in the function or if that number causes us to try to calculate the square root of a negative number.

Problems

Determine the domain of each of the following functions.

1. $f(x) = -3x + 5$

2. $f(x) = \sqrt{2x + 4}$

3. $f(x) = \dfrac{x+2}{x-3}$

4. $f(x) = \dfrac{1}{\sqrt{x-5}}$
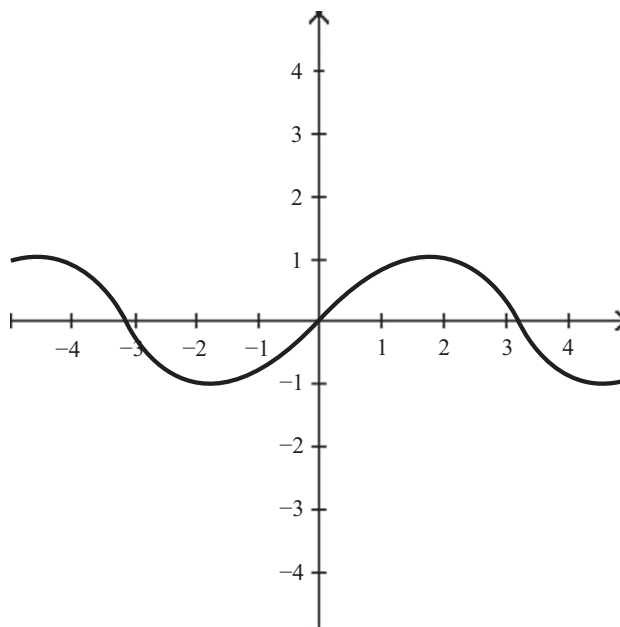
Find $f(g(x))$ and $g(f(x))$ for each of the following.

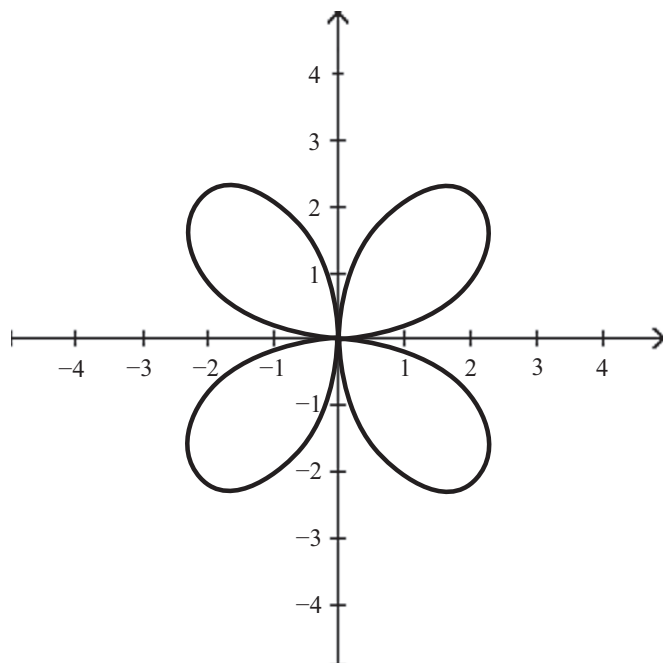5. $f(x) = 2x + 1;\ g(x) = 5x - 1$

6. $f(x) = x^2;\ g(x) = 2x$

7. $f(x) = -x + 1;\ g(x) = x - 1$

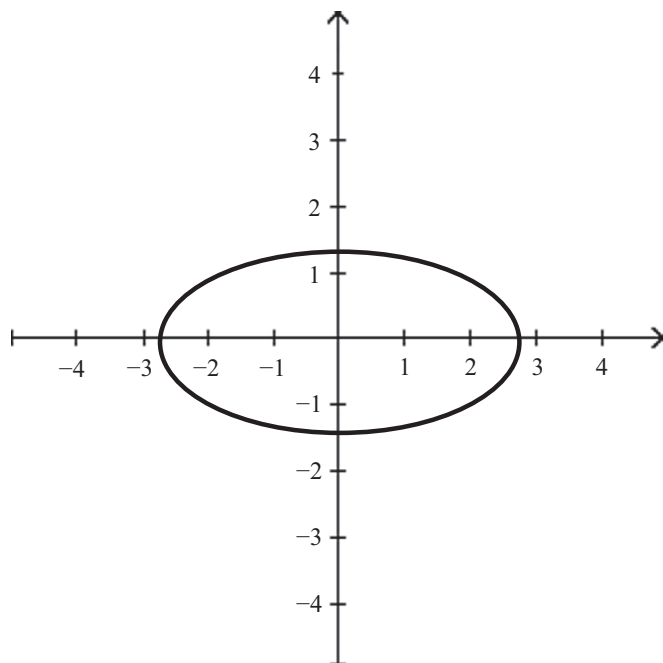Using the vertical line test, determine whether the following graphs represent functions.

8.

9.

10.

Systems of 2 Linear Equations, Part 1
Lesson 7

Topics in This Lesson

- The concept of a system of linear equations and what a solution of such a system means.

- Envisioning a solution of a system of linear equations by graphing.

- Using graphing to solve a system of linear equations.

Summary

A set of 2 or more linear equations forms a system of linear equations. An ordered pair that makes all of the equations in a system of linear equations true is a solution of the system. From a graphical perspective, a solution of a system of equations is an intersection point between all the graphs of the equations in the system. In this lesson, we learn how to find solutions of systems of linear equations by graphing. We also discuss some of the positives, as well as the negatives, to using this approach.

Definitions and Formulas

dependent system: A system of linear equations that has infinitely many solutions (because the 2 lines in the system are actually the same line).

inconsistent system: A system of linear equations that has no solution.

independent system: A system of linear equations that has a unique solution.

solution of a system: An ordered pair that makes all of the equations in a system of linear equations true.

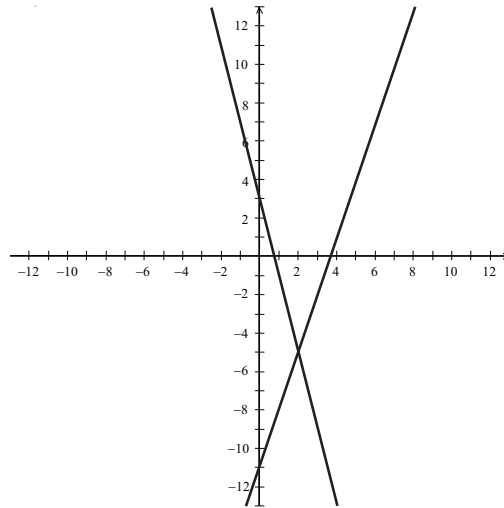**system of linear equations**: A set of 2 or more linear equations.

Examples

Example 1

Solve the following system by graphing.

$$y = -4x + 3$$
$$y = 3x - 11$$

We begin by drawing the lines represented by these 2 equations (on the same set of axes).

Notice that the intersection point appears to be (2, −5). This means our solution is (2, −5). Let's confirm this.

$$y = -4x + 3$$
$$-5 = -4(2) + 3$$
$$-5 = -8 + 3$$
$$-5 = -5$$

This is true.

$$y = 3x - 11$$
$$-5 = 3(2) - 11$$
$$-5 = 6 - 11$$
$$-5 = -5$$

This is also true. So, (2, −5) is a solution of the system.

Example 2

Solve this system by graphing.

$$3x + 2y = 5$$
$$6x + 4y = 19$$

We see that the equations in our system are in standard form. In order to graph them, we should write them in either point-slope or slope-intercept form. Slope-intercept form is the most straightforward to use in this case.
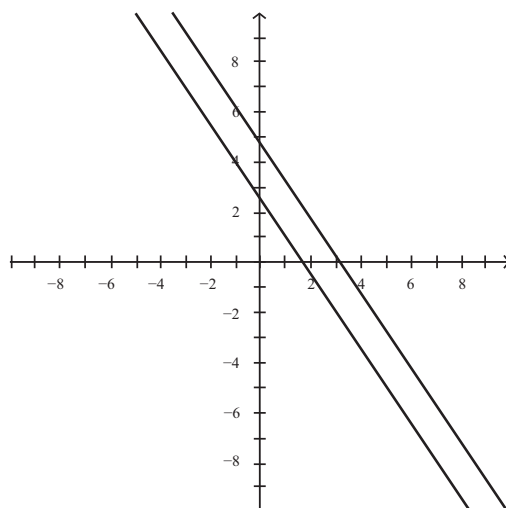
$$3x + 2y = 5$$
$$2y = -3x + 5$$
$$y = -3/2x + 5/2$$

$$6x + 4y = 19$$
$$4y = -6x + 19$$
$$y = -3/2x + 19/4$$

So, our original system is equivalent to

$$y = -3/2 \ x + 5/2$$
$$y = -3/2 \ x + 19/4.$$

Notice that these lines have the same slope but different *y*-intercepts. That means they are parallel. Here's a sketch of the graphs.

This means there is no solution to this system, because parallel lines never intersect. This is known as an inconsistent system.

Example 3

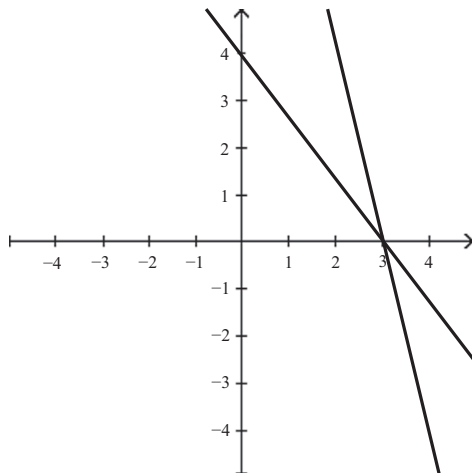Solve this system of linear equations by graphing.

$$4x + 3y = 12$$
$$6x + 2y = 18$$

Since we need to draw the graphs of the equations in order to look for the intersection point, we first rewrite the equations in slope-intercept form. This will make sketching the graph more straightforward.

$$4x + 3y = 12$$
$$3y = -4x + 12$$
$$y = -\frac{4}{3}x + 4$$

$$6x + 2y = 18$$
$$2y = -6x + 18$$
$$y = -3x + 9$$

Graphing these 2 lines gives us the following.

It appears to be the case that (3, 0) is the solution. Let's confirm this.

$$4x + 3y = 12$$
$$4(3) + 3(0) = 12$$
$$12 + 0 = 12$$
$$12 = 12$$

This is true.

$$6x + 2y = 18$$
$$6(3) + 2(0) = 18$$
$$18 + 0 = 18$$
$$18 = 18$$

This is also true.

So we have confirmed that $x = 3$, $y = 0$ really is the solution of this system of linear equations.

Common Errors

- At times, it can be difficult to see what the solution of a system is based on the graphs of the linear equations.

- Draw the graphs of these lines as carefully as possible; this method requires you to see where the intersection point is between these 2 graphs, so accuracy is very important.

Study Tips

- When sketching the graphs of functions, use graph paper or grid paper.

- Students often skip the step of checking their solutions. You should always plug your solution into the original set of equations, especially when you are finding the solution by graphing.

Problems

Find the solution of each system of linear equations.

1. $y = 5x - 6$
    $y = -5x + 4$

2. $-4x + 3y = 9$
    $2x + 3y = -9$

3. $y = \dfrac{2}{3}x + 1$
    $y = \dfrac{2}{3}x - 4$

4. $y = -x - 2$
    $-x + 3y = 6$

5. $3x + 2y = -8$
    $-x + 2y = 8$

6. $y = 2x - 1$
    $y = -x + 2$

7. $y = \dfrac{4}{9}x + 5$

$y = \dfrac{2}{7}x + 5$

8. $y = -\dfrac{1}{2}x + 1$

$y = -\dfrac{3}{2}x - 3$

9. $x - y = 4$
$x + y = -8$

10. $y = 3x + 4$
$y = -2x + 14$

Topics in This Lesson

- Solving systems of linear equations with the method of substitution.

- Solving systems of linear equations with the method of elimination.

Summary

In the previous lesson, we introduced the idea of a solution of a system of linear equations, and we talked about one way to find such a solution—by graphing the lines that correspond to the linear equations in the system and looking for the intersection point. Unfortunately, the method of solving by graphing has weaknesses, so we need other methods for solving systems of linear equations. In this lesson, we introduce 2 such methods—the method of substitution and the method of elimination.

Examples

Example 1

Solve this system.

$$2x + 3y = 4$$
$$-6x + y = -7$$

We add $6x$ to both sides of the second equation to get a new equation: $y = 6x - 7$.

Now we can replace the y in the first equation with $6x - 7$ (since $y = 6x - 7$). This is our substitution. Then the first equation becomes $2x + 3(6x - 7) = 4$.

Now we need to solve for x. (Notice that there are no y's in this new equation.)

$$2x + 3(6x - 7) = 4$$
$$2x + 3(6x) - 3(7) = 4$$
$$2x + 18x - 21 = 4$$
$$20x - 21 = 4$$
$$20x = 25$$
$$x = \frac{25}{20} = \frac{5}{4}$$

Remember that $y = 6x - 7$. Using $x = \dfrac{5}{4}$, we see that

$$y = 6\left(\dfrac{5}{4}\right) - 7$$

$$= 3\left(\dfrac{5}{2}\right) - 7$$

$$= \dfrac{15}{2} - 7$$

$$= \dfrac{15}{2} - \dfrac{14}{2}$$

$$= \dfrac{1}{2}.$$

So our solution to this system is $x = \dfrac{5}{4}, y = \dfrac{1}{2}$, or the ordered pair $\left(\dfrac{5}{4}, \dfrac{1}{2}\right)$.

Let's check to make sure $\left(\dfrac{5}{4}, \dfrac{1}{2}\right)$ is a solution to the original system.

$$2x + 3y = 4$$

$$2\left(\dfrac{5}{4}\right) + 3\left(\dfrac{1}{2}\right) = 4$$

$$\dfrac{5}{2} + \dfrac{3}{2} = 4$$

$$\dfrac{8}{2} = 4$$

$$4 = 4$$

$$-6x + y = -7$$

$$-6\left(\dfrac{5}{4}\right) + \dfrac{1}{2} = -7$$

$$-3\left(\dfrac{5}{2}\right) + \dfrac{1}{2} = -7$$

$$-\dfrac{15}{2} + \dfrac{1}{2} = -7$$

$$-\dfrac{14}{2} = -7$$

$$-7 = -7$$

Example 2

Solve this system by elimination.

$$2x - 5y = 12$$
$$6x + 5y = 36$$

First, notice that if we add the left-hand sides, we have the following.

$$\begin{array}{r} 2x - 5y \\ + \; 6x + 5y \\ \hline 8x + 0y \end{array}$$

This is the same as just $8x$.

If we then add the right-hand sides, we have the following.

$$\begin{array}{r} 2x - 5y = 12 \\ + \; 6x + 5y = 36 \\ \hline 8x + 0 = 48 \end{array}$$

The x-value that will make this new equation true will also be the x-value in the solution of our original system. So we solve that equation for x.

$$8x/8 = 48/8$$
$$x = 6$$

So we see that $x = 6$. But we still need y. Choose either of the original equations, plug in 6 for x, and solve for y, as follows.

$$2x - 5y = 12$$
$$2(6) - 3y = 12$$
$$12 - 3y = 12$$
$$-3y = 0$$
$$y = 0$$

So the solution of the system appears to be (6, 0). Let's check.

$$2x - 5y = 12$$
$$2(6) - 3(0) = 12$$
$$12 - 0 = 12$$
$$12 = 12$$

$$6x + 5y = 36$$
$$6(6) + 3(0) = 36$$
$$36 + 0 = 36$$
$$36 = 36$$

Example 3

Solve this system of linear equations using elimination.

$$-2x + 15y = -33$$
$$-x + 3y = 6$$

We multiply the second equation in the original system by −5. This gives the following equivalent system.

$$-2x + 15y = -33$$
$$5x - 15y = -30$$

Adding these 2 equations gives us the following.

$$3x + 0 = -63$$
$$3x = -63$$

$$(-21, -5)$$

Dividing both sides of this equation by 3 gives $x = -21$. We plug $x = -21$ back into one of the original equations to find y.

$$-2x + 15y = -33$$
$$-2(-21) + 15y = -33$$
$$42 + 15y = -33$$
$$15y = -75$$
$$y = -\frac{75}{15} = -5$$

The final solution is $(-21, -5)$.

Check it.

$$-2x + 15y = -33$$
$$-2(-21) + 15(-5) = -33$$
$$42 - 75 = -33$$
$$-33 = -33$$

$$-x + 3y = 6$$
$$-(-21) + 3(-5) = 6$$
$$21 - 15 = 6$$
$$6 = 6$$

Common Errors

- Not being careful with the arithmetic in these problems. Be sure to watch for distributive property errors, and remember to add or subtract throughout when eliminating.

Study Tips

- Take your time in performing your arithmetic, and be sure to write out all steps.

- Check your solutions by plugging them into the original equations, just to be sure everything is correct.

Problems

Solve the following systems of linear equations using substitution.

1. $y = 2x + 7$
    $y = 4x - 19$

2. $y = -5x + 3$
    $x + y = 27$

3. $y = 5x - 7$
    $3x + 2y = 12$

4. $-5x + y = -2$
    $3x - 6y = 12$

5. $-3x + 3y = 4$
    $x - y = -3$

Solve the following systems of linear equations using elimination.

6. $x - y = 13$
    $2x + y = 17$

7. $7x + 2y = 22$
    $8x + 2y = 32$

8. $-4x + 4y = 18$
    $-4x + 2y = 12$

9. $x - 4y = 3$
    $-5x + 20y = -15$

10. $4x + 2y = -14$
    $10x - 7y = 25$

Systems of 3 Linear Equations
Lesson 9

Topics in This Lesson

- Solving systems of linear equations with 3 variables.

Summary

In the last few lessons, we talked about solving systems of linear equations, and we spent most of our time on systems with 2 linear equations and 2 unknowns (or variables). In this lesson, we discuss systems of linear equations with 3 equations and 3 unknowns.

Examples

Example 1

Solve this system of 3 linear equations.

$$2x + 3y - z = 15$$
$$x - 3y + 3z = -4$$
$$4x - 3y - z = 19$$

We want to choose pairs of equations that can be combined to eliminate one of the variables. Once we have eliminated the same variable twice, we will be left with a system of 2 linear equations with 2 unknowns, and we know how to solve that smaller system. We will solve that smaller system, and we will almost be finished.

We focus our attention on the first and second equations.

$$2x + 3y - z = 15$$
$$x - 3y + 3z = -4$$

If we add these 2 equations, we have

$$3x + 2z = 11.$$

Notice that the variable y has been eliminated. Next, we consider the first and third equations.

$$2x + 3y - z = 15$$
$$4x - 3y - z = 19$$

We can add them together to eliminate the y's.

$$6x - 2z = 34$$

So we now have a system of 2 equations with 2 unknowns.

$$3x + 2z = 11$$
$$6x - 2z = 34$$

Notice that adding these 2 equations will then eliminate the variable z.

$$9x = 45$$
$$x = 5$$

We now know the value of x in our solution: $x = 5$. We can put $x = 5$ back into either $3x + 2z = 11$ or $6x - 2z = 34$, and we can then solve for z.

$$3x + 2z = 11$$
$$3(5) + 2z = 11$$
$$15 + 2z = 11$$
$$2z = -4$$
$$z = -2$$

So now we know that $x = 5$ and $z = -2$. We can go back to any of the 3 original equations and plug in $x = 5$ and $z = -2$ to find y.

$$2x + 3y - z = 15$$
$$2(5) + 3y - (-2) = 15$$
$$10 + 3y + 2 = 15$$
$$3y + 12 = 15$$
$$3y = 3$$
$$y = 1$$

We now have our final solution: $x = 5$, $y = 1$, $z = -2$. Notice that in this example, we found x, then z, then y. The order in which we find the values of the variables does not matter. You do not have to find x, then y, then z, for example. You can find the values in whatever order works best.

Common Errors

- It is easy to make errors in the arithmetic in these problems. Watch for distributive property errors, and remember to add or subtract throughout when eliminating.

- Misinterpreting contradictory information, like $0 = -5$. You may take this to mean that you have made an arithmetic mistake in your calculations. Remember: It is possible that the system is inconsistent (that it has no solution), and an equation like $0 = -5$ is just an indication of this.

Study Tips

- Take your time in performing your arithmetic, and be sure to write out all steps.

- The order in which you find the values of the variables does not matter. You can find the values in whatever order works best.

- Check your solutions by plugging them into the original equations, just to be sure everything is correct.

Problems

Solve the following systems.

1. $x + 2y + 3z = 9$
    $2x - y + z = 8$
    $3x - z = 3$

2. $x - 3y = 2$
    $x + 2y + 5z = 2$
    $-2x + 6y + 4z = 4$

3. $x - y = -1$
    $x + y + z = 1$
    $-3x - z = 5$

4. $x - y + z = 7$
    $3x + y + 6z = 1$
    $-2x + 2y - 2z = 5$

5. $x + y + z = 6$
    $x - y - z = 0$
    $2x - 3y + 5z = 5$

Solving Systems of Linear Inequalities
Lesson 10

_____

Topics in This Lesson

- Graphing the solution set of one inequality.

- Solving systems of linear inequalities.

Summary

In this lesson, we discuss what the solution set of an inequality looks like (from a graphical perspective), and we also consider what the solution of a system of linear inequalities can be. The solution set is the intersection of the solution sets of each individual linear inequality.

Definitions and Formulas

strict inequality: An inequality that is only greater than or less than (> or <), not greater than or equal to ($\geq$) or less than or equal to ($\leq$).
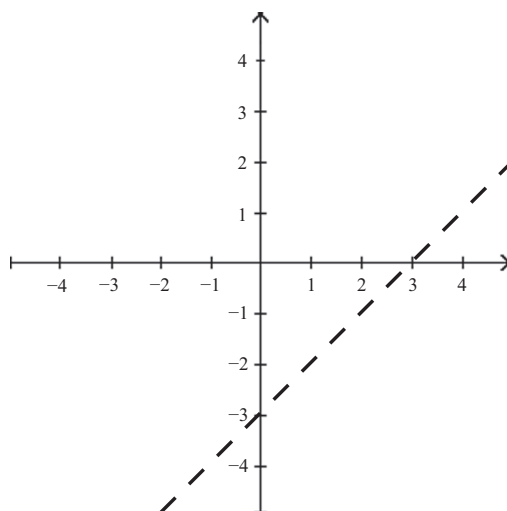
Examples

Example 1

Solve the system of inequalities given by

$$y > x - 3$$
$$2x + y < 4.$$

The first thing we notice is that both inequalities are strict (one is greater than and the other is less than). So when we draw the boundary lines in this problem, they will both be drawn as dashed lines.
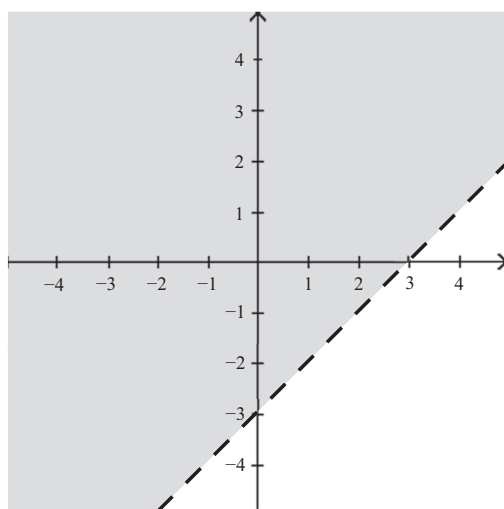
We begin by drawing the first line, $y = x - 3$.

In this case, we shade in those points satisfying $y > x - 3$. Let's check to see if the origin is in the set.

$$y > x - 3$$
$$0 > 0 - 3$$
$$0 > -3$$

This statement is true, so the origin is in the set. This means we shade the part of the plane that is above and to the left of the line.
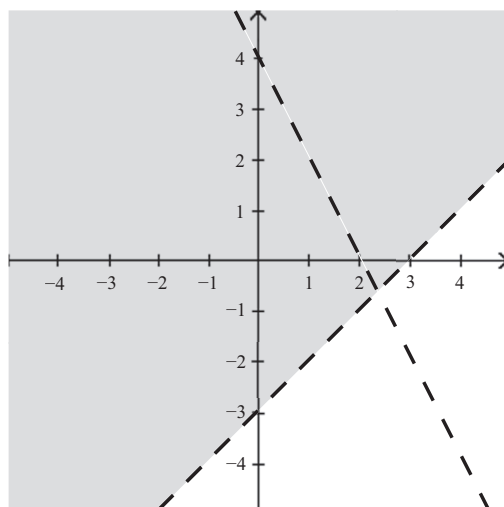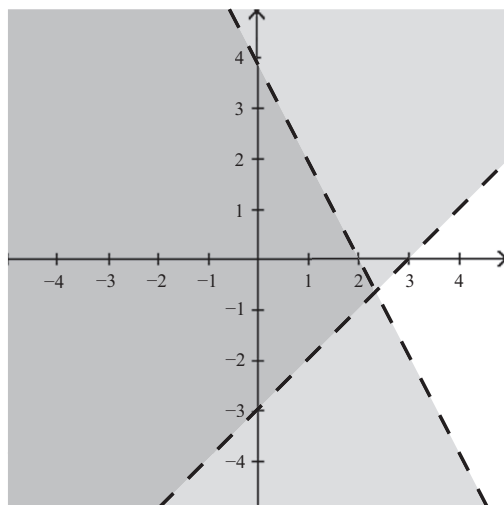
Next, we move to the second inequality, $2x + y < 4$. We need to draw the boundary line related to this inequality, which would be $2x + y = 4$.

$$2x + y = 4$$
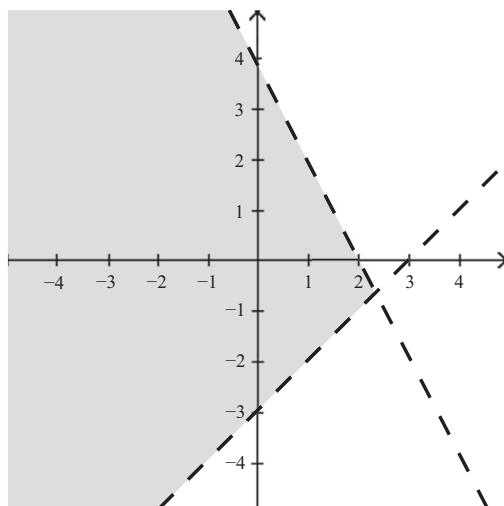$$y = -2x + 4$$

Now we shade in the correct region of the plane that corresponds to the inequality $y < -2x + 4$. These are the points that are below and to the left of the line.

47

The solution set is represented by the intersection of the 2 shaded regions. Every point in that region satisfies the system of linear inequalities.
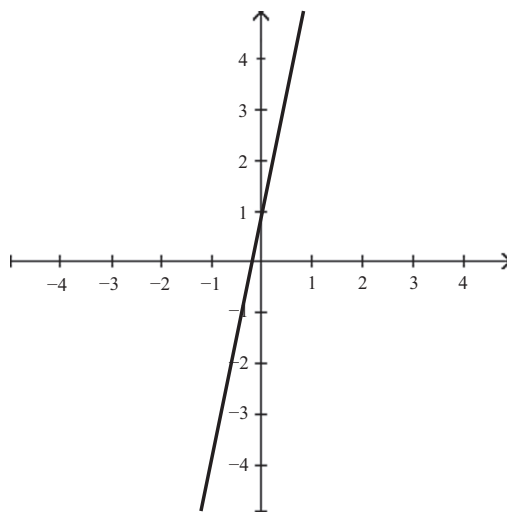

Example 2

Solve the system of linear inequalities given by the following.

$$y \leq 5x + 1$$
$$y \geq 2x - 3$$

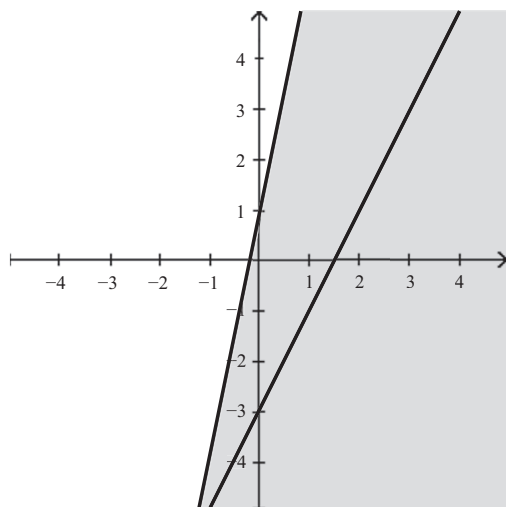We begin by graphing the boundary line for the first inequality, $y = 5x + 1$. This is drawn as a solid line since the inequality is greater than or equal to.
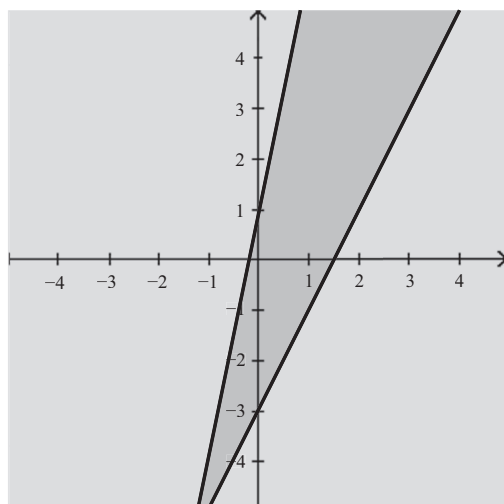
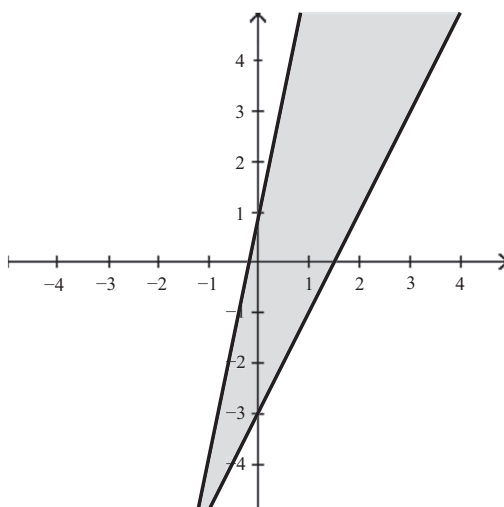We then shade in the solution set for $y \le 5x + 1$.
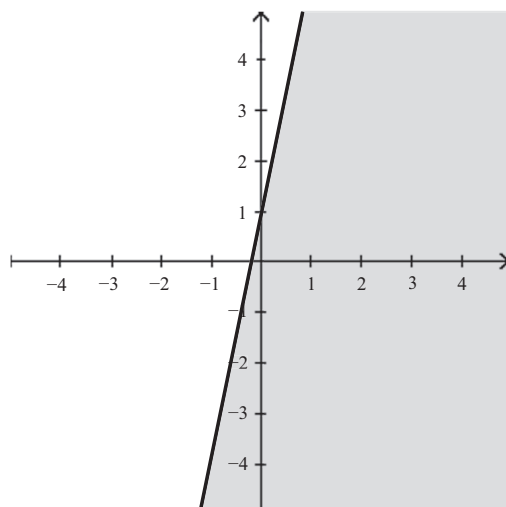
Next, we draw in the boundary line for the other inequality.

We then shade in the solution set for the inequality $y \geq 2x - 3$.

The solution set is just the intersection as shown below.

Common Errors

- Selecting the wrong portions of the plane to shade when determining solution sets.

Study Tips

- Be careful to remember to draw dashed boundary lines when dealing with strict inequalities. It is tempting to always draw solid lines.

- Choose a point in the plane, like (0, 0), and see if it satisfies the given inequality. If it does, then shade in the portion of the plane that contains (0, 0). If this point does not satisfy the given inequality, then shade in the portion of the plane that does not contain this point.

Problems

Solve the following systems of linear inequalities.

1. $y < x - 2$
    $y \geq 2x - 3$

2. $y > 4x$
    $y > 2$

3. $\dfrac{x^a}{x^b} = x^{a-b}$

4. $y \leq -3x + 7$
    $y \geq 2x - 6$

5. $y \leq 4$
    $y \leq -3x + 2$

6. $y < -2x - 4$
    $y < 6x + 2$

7. $y \geq -x$
    $y \geq 2x$

8. $y > -2$
    $y \geq 3$

9. $y \leq 4x - 3$
    $y \geq -3x + 7$

10. $y > 5x$
    $y < 2x$

An Introduction to Quadratic Functions
Lesson 11

Topics in This Lesson

- Quadratic functions.

- Parabolas—the graphs of quadratic functions.

Summary

In this lesson, we introduce quadratic functions (which are of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers). We talk about the graphs of these kinds of functions, which are known as parabolas, and discuss the different shapes that can occur.

Definitions and Formulas

parabola: The graph of a quadratic function.

**quadratic function**: A function of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers.

vertex of the parabola: The lowest point of a U-shaped parabola or the highest point of an upside-down U-shaped parabola.

Examples

Example 1

Sketch the graph of $h(x) = (x-2)^2$.

Remember that the graph of $h(x)$ is very similar to the graph of x^2; it just shifts to the right 2 units. So the graph looks like this.
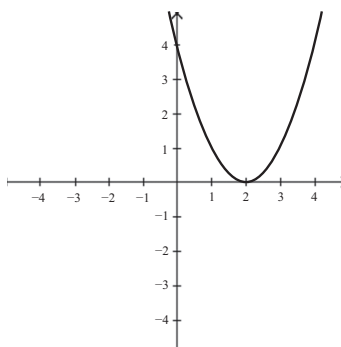
Example 2

Sketch the graph of $f(x) = (x+3)^2 - 4$.

To sketch this graph, we start with the graph of x^2, shift that graph 3 units to the left, and then shift that graph down 4 units. The graph of $f(x)$ then looks like this.
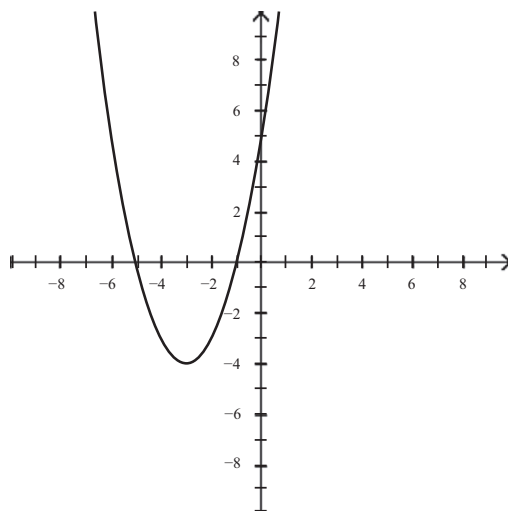
Common Errors

- When replacing x with $(x - a)$ for some number a, it is very tempting to shift the graph to the left rather than the right.

Study Tips

- When drawing graphs of functions, it is a good idea to use graph or grid paper.

Problems

Sketch the graph of each of the following quadratic functions.

1. $f(x) = x^2 - 5$

2. $f(x) = (x - 4)^2$

3. $f(x) = (x + 1)^2$

4. $f(x) = x^2 + 2$

5. $f(x) = (x-3)^2 - 1$

6. $f(x) = (x+2)^2 + 3$

7. $f(x) = (x+4)^2 - 5$

8. $f(x) = -x^2$

9. $f(x) = -x^2 + 2$

10. $f(x) = -(x-3)^2$

Quadratic Equations—Factoring
Lesson 12

Topics in This Lesson

- Factoring quadratic polynomials by using greatest common factors.

- Factoring quadratic polynomials by using the opposite of FOIL.

- Solving quadratic equations with factoring.

Summary

One of the most important topics in algebra is solving equations. In this lesson, we consider how to solve quadratic equations using the concept of factoring.

Definitions and Formulas

x-intercepts: The points where the graph crosses the *x*-axis.

Examples

Example 1

Determine the *x*-intercepts of the parabola whose equation is $f(x) = 10x^2 - 8x$.

The *x*-intercepts of any graph are the points where the graph crosses the *x*-axis. That means these points have a *y*-coordinate of 0. So we set the function equal to 0 and solve for *x*.

$$10x^2 - 8x = 0$$

This factors as follows.

$$10x^2 - 8x = 0$$
$$2x(5x - 4) = 0$$

Now we use a very important rule: If you multiply 2 real numbers together, and their product equals 0, then one (or both) of the numbers you multiplied together must have been 0. So if $2x(5x-4)=0$, then $2x=0$ or $5x-4=0$.

$$2x = 0$$
$$x = 0$$

$$5x - 4 = 0$$
$$5x = 4$$
$$x = 4/5$$

So we now have our answer: The *x*-intercepts of this parabola are the points (0, 0) and (4/5, 0).

Example 2

Solve $2x^2 + 8x - 20 = 5x$.

We first move all the terms to the left-hand side of the equation.

$$2x^2 + 8x - 20 - 5x = 0$$
$$2x^2 + 3x - 20 = 0$$

After attempting several possibilities for the factorization, we see that the factorization is $(2x - 5)(x + 4)$. (This can be checked by expanding this product using the FOIL process.)

Now we can finish finding the solutions.

$$2x^2 + 3x - 20 = 0$$
$$(2x - 5)(x + 4) = 0$$

$$2x - 5 = 0 \text{ or } x + 4 = 0$$
$$2x = 5 \text{ or } x = -4$$
$$x = \frac{5}{2} \text{ or } x = -4$$

These are our solutions.

Example 3

Solve the quadratic equation $t^2 - 14 = 5t$.

The first thing we do is move all the terms to the left-hand side of the equation.

$$t^2 - 14 = 5t$$
$$t^2 - 5t - 14 = 0$$

We know that the factored form of the quadratic polynomial on the left-hand side must be of the form $(t +$ something$)(t -$ something$)$. The signs have to be different so that the product of those 2 numbers is negative (the product must equal -14). We now need to insert 2 numbers whose product is -14. The correct choice is

$$(t - 7)(t + 2).$$

So the equation we wish to solve can be rewritten as

$$(t - 7)(t + 2) = 0.$$

This means $t - 7 = 0$ or $t + 2 = 0$. These can be rewritten as $t = 7$ or $t = -2$. These are the 2 solutions.

Common Errors

- When FOILing the product of 2 binomials to check whether the factoring of a quadratic polynomial is correct, some students make arithmetic errors (especially when some of the terms are added and some are subtracted within the binomials).

Study Tips

- Take your time when trying to guess the numbers to use when factoring a quadratic polynomial. This can take some time, so be patient.

Problems

Factor the following quadratic polynomials.

1. $x^2 + 8x + 15$

2. $y^2 + 2y - 24$

3. $x^2 - x - 42$

4. $x^2 - 14x + 40$

5. $5t^2 - 30t + 40$

Solve the following quadratic equations.

6. $x^2 + 18 = 11x - 6$

7. $x^2 + 24 = 10x$

8. $x^2 = 18 - 3x$

9. $3x^2 - 12x - 7 = 4x + 5$

10. $x^2 - 10x = -25$

Quadratic Equations—Square Roots
Lesson 13

Topics in This Lesson

- The set of complex numbers.

- Solving quadratic equations by using square roots.

Summary

In this lesson, we discuss solving quadratic equations by using square roots. We also introduce the idea of complex numbers, which include the imaginary number $i = \sqrt{-1}$.

Definitions and Formulas

$i = \sqrt{-1}$

Examples

Example 1

Solve the equation $x^2 - 144 = 0$.

$$x^2 - 144 = 0$$
$$x^2 = 144$$
$$x = \pm\sqrt{144}$$
$$x = \pm 12$$

So the solutions are $x = 12$ and $x = -12$. We can check those quickly just to make sure.

$$x^2 - 144 = 0$$
$$(12)^2 - 144 = 0$$
$$144 - 144 = 0$$
$$0 = 0$$

$$x^2 - 144 = 0$$
$$(-12)^2 - 144 = 0$$
$$144 - 144 = 0$$
$$0 = 0$$

Example 2

Solve $(x-3)^2 - 49 = 0$.

$$(x-3)^2 - 49 = 0$$
$$(x-3)^2 = 49$$
$$\sqrt{(x-3)^2} = \pm\sqrt{49}$$
$$x - 3 = \pm 7$$

So

$$
\begin{array}{ll}
x - 3 = 7 & x - 3 = -7 \\
x = 7 + 3 \quad \text{or} & x = -7 + 3 \\
x = 10 & x = -4.
\end{array}
$$

Let's check to make sure these are the correct solutions.

$$(x-3)^2 - 49 = 0$$
$$(10-3)^2 - 49 = 0$$
$$7^2 - 49 = 0$$
$$49 - 49 = 0$$
$$0 = 0$$

$$(x-3)^2 - 49 = 0$$
$$(-4-3)^2 - 49 = 0$$
$$(-7)^2 - 49 = 0$$
$$49 - 49 = 0$$
$$0 = 0$$

Example 3

Solve $(x-6)^2 + 81 = 0$.

$$(x-6)^2 + 81 = 0$$
$$(x-6)^2 = -81$$
$$\sqrt{(x-6)^2} = \pm\sqrt{-81}$$
$$x - 6 = \pm\sqrt{-81}$$

At this point, we see the square root of a negative number on the right-hand side of the equation. This means that there are no real number solutions. However, if we are allowed to have complex number solutions, we can proceed.

$$x - 6 = \pm\sqrt{-81}$$
$$x - 6 = \pm 9i$$
$$x = 6 \pm 9i$$

So our complex number solutions are $6 + 9i$ and $6 - 9i$.

Common Errors

- Forgetting to insert the plus or minus symbol as you solve these equations. If this symbol is not included, then some solutions may be missed.

Study Tips

- Review your square root facts if needed; knowing the values of several square roots will help in solving these equations.

Problems

Solve the following quadratic equations.

1. $x^2 - 169 = 0$

2. $x^2 + 36 = 0$

3. $x^2 - 3 = 61$

4. $4x^2 = 25$

5. $7x^2 - 1 = 27$

6. $8x^2 - 23 = 177$

7. $(x + 7)^2 - 9 = 0$

8. $(2x - 3)^2 - 225 = 0$

9. $9(x + 10)^2 + 121 = 0$

10. $4(x - 2)^2 - 289 = 0$

Completing the Square
Lesson 14

Topics in This Lesson

- How to complete the square.

- Using completing the square to quickly identify the vertex of a parabola.

- Using completing the square to solve quadratic equations.

Summary

We continue to discuss tools that can be used to solve quadratic equations. In this lesson, we discuss the tool of completing the square. In the process, we show how to use completing the square to help in identifying the vertex of a given parabola and to solve a quadratic equation.

Definitions and Formulas

In order to **complete the square** for a quadratic function of the form $f(x) = ax^2 + bx + c$, where b and c are real numbers, we must both add and subtract the quantity $(b/2)^2$, or $b^2/4$, on the polynomial side.

Examples

Example 1

Complete the square for the quadratic function $f(x) = x^2 - 10x - 30$.

Take the coefficient in front of the x (in this case 10), ignoring the sign in front of it, divide it by 2, and then square that new number. In this case, $10/2 = 5$ and $5^2 = 25$. Therefore, to complete the square, we add and subtract 25. Then the original expression becomes

$$x^2 - 10x + 25 - 25 - 30$$
$$= (x^2 - 10x + 25) - 55$$
$$= (x - 5)^2 - 55.$$

Therefore, $f(x) = (x - 5)^2 - 55$ is our final answer.

Example 2

Solve the equation $-x^2 + 8x = -40$.

First, multiply both sides of the equation by -1.

$$
\begin{aligned}
-x^2 + 8x &= -40 \\
-1(-x^2 + 8x) &= -1(-40) \\
x^2 - 8x &= 40 \\
x^2 - 8x + 16 - 16 &= 40 \\
(x^2 - 8x + 16) - 16 &= 40 \\
(x-4)^2 - 16 &= 40 \\
(x-4)^2 &= 56
\end{aligned}
$$

Taking square roots gives us the following.

$$x - 4 = \pm\sqrt{56}$$

This means $x - 4 = \sqrt{56}$ or $x - 4 = -\sqrt{56}$. Therefore, the 2 solutions are $x = 4 + \sqrt{56}$ and $x = 4 - \sqrt{56}$. These can be simplified slightly to $x = 4 + 2\sqrt{14}$ and $x = 4 - 2\sqrt{14}$, since $56 = 4 \times 14$.

Example 3

Find the coordinates of the vertex of the parabola whose equation is $f(x) = x^2 - 12x + 47$.

We begin by rewriting $f(x)$ by completing the square.

$$
\begin{aligned}
&x^2 - 12x + 47 \\
={}& x^2 - 12x + 36 - 36 + 47 \\
={}& (x-6)^2 - 36 + 47 \\
={}& (x-6)^2 + 11
\end{aligned}
$$

So $f(x) = (x-6)^2 + 11$. Therefore, the coordinates of the vertex of this parabola are $x = 6$ and $y = 11$, so the vertex is at (6, 11).

Common Errors

- Confusing the coordinates of the vertex of the parabola, especially in terms of the signs of the 2 values. For example, in the last example just completed, you might say that the vertex is at (−6, 11) rather than (6, 11).

Study Tips

- Be very careful with the arithmetic in these problems, especially if the coefficient of the linear term (the *x* term) is odd.

Problems

Find the vertex of each parabola given by the following quadratic functions.

1. $f(x) = x^2 - 20x + 103$

2. $f(x) = x^2 + 14x + 32$

3. $f(x) = x^2 + 2x - 7$

4. $f(x) = x^2 - 16x + 64$

5. $f(x) = x^2 + 3$

Solve the following quadratic equations using completing the square.

6. $x^2 - 6x = -5$

7. $x^2 - 7x = 3x - 9$

8. $x^2 - 12x = -18x - 16$

9. $3x^2 - 6x = 2x^2 - 7$

10. $-x^2 + 16x = -36$

Using the Quadratic Formula
Lesson 15

Topics in This Lesson

- Definition of the quadratic formula.

- Solving equations by using the quadratic formula.

- Definition of the discriminant.

- Using the sign of the discriminant to quickly determine the number of real number solutions of a quadratic equation.

Summary

We have seen a number of tools for solving quadratic equations. In this lesson, we learn another such tool that is more versatile than the others (but is also viewed as more cumbersome by some students). That tool is called the quadratic formula. Part of the quadratic formula is the discriminant, and we see in this lesson how the discriminant can be used to quickly tell how many real number solutions a quadratic equation has.

Definitions and Formulas

For a quadratic equation of the form $ax^2 + bx + c = 0,$ the solutions can be found by using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

discriminant: The quantity $b^2 - 4ac$ for the quadratic equation $ax^2 + bx + c = 0.$

Examples

Example 1

Solve $x^2 - 7x - 14 = 5x + 10.$

First, we move all the terms over to the left-hand side of the equation and combine like terms.

$$x^2 - 12x - 24 = 0$$

Now we apply the quadratic formula with $a = 1$, $b = -12$, $c = -24$. Then the solutions of the equation are

$$x = \frac{-(-12)+\sqrt{(-12)^2 - 4(1)(-24)}}{2(1)}$$

and

$$x = \frac{-(-12)-\sqrt{(-12)^2 - 4(1)(-24)}}{2(1)}.$$

We can simplify these quickly.

$$x = \frac{-(-12)+\sqrt{(-12)^2 - 4(1)(-24)}}{2(1)}$$

$$= \frac{12+\sqrt{144+96}}{2}$$

$$= \frac{12+\sqrt{240}}{2}$$

$$= \frac{12+\sqrt{16 \cdot 15}}{2}$$

$$= \frac{12+4\sqrt{15}}{2}$$

$$= 6+2\sqrt{15}$$

The other solution simplifies in similar fashion and is equal to $6-2\sqrt{15}$.

Example 2

Solve the equation $7x^2 +10x+8 = 4x^2 + 2x +11$.

Let's start by moving all the terms to the left side of the equation and combining like terms. Then we have

$$3x^2 + 8x - 3 = 0.$$

Using the quadratic formula, $a = 3$, $b = 8$, and $c = -3$. That means the solutions are as follows.

$$x = \frac{-(8) + \sqrt{(8)^2 - 4(3)(-3)}}{2(3)}$$

$$x = \frac{-8 + \sqrt{64 - (-36)}}{6}$$

$$x = \frac{-8 + \sqrt{100}}{6}$$

$$x = \frac{(-8) + (10)}{6}$$

$$x = \frac{2}{6}$$

$$x = \frac{1}{3}$$

and

$$x = \frac{-(8) - \sqrt{(8)^2 - 4(3)(-3)}}{2(3)}$$

$$x = \frac{-8 - \sqrt{64 - (-36)}}{6}$$

$$x = \frac{-8 - \sqrt{100}}{6}$$

$$x = \frac{(-8) - (10)}{6}$$

$$x = \frac{-18}{6}$$

$$x = -3$$

So our 2 solutions are $x = \dfrac{1}{3}$ and $x = -3$.

Common Errors

- Losing track of all the different plus and minus signs in the formula. Make sure that you copy them correctly after each step.

- Forgetting that there are 2 different possible solutions for each quadratic equation according to the quadratic formula. Don't forget to substitute in for both formulas!

Study Tips

- Neat handwriting will really make a difference with your math work. The more complicated a problem, the more likely you are to make an error. Do yourself a favor—write in tidy columns with neat handwriting.

- You can use a calculator to find square roots. However, it's a great idea to memorize the first 20 squares and square roots.

Problems

Solve the following equations.

1. $x^2 + 9x + 20 = 0$

2. $x^2 - 8x + 16 = 0$

3. $5x^2 + 11x + 2 = 0$

4. $6x^2 - 17x + 9 = 4x^2 - 14x + 3$

5. $-x^2 - 4x + 7 = 0$

Use the discriminant to determine whether the following equations have 0, 1, or 2 real number solutions.

6. $x^2 + 6x + 9 = 0$

7. $x^2 + 7x - 13 = 0$

8. $x^2 + 9x + 21 = 0$

9. $3x^2 + 9x - 5 = 0$

10. $-x^2 + 3x - 5 = 0$

Solving Quadratic Inequalities
Lesson 16

Topics in This Lesson

- The definition of a quadratic inequality.

- Solving quadratic inequalities.

- Solving systems of quadratic inequalities.

Summary

In this lesson, we consider inequalities that contain quadratic terms (terms that are squared). Such inequalities are called quadratic inequalities. We learn how to graph the solutions of individual quadratic inequalities and how to solve systems of quadratic inequalities.

Definitions and Formulas

quadratic inequality: An inequality that involves at least one term that is raised to the second power.

Examples

Example 1

Graph the solution set for $y \geq x^2 - 10x + 21$.

We begin by graphing the boundary parabola $y = x^2 - 10x + 21$. Notice that the right-hand side factors nicely as $(x - 7)(x - 3)$.

That means there are 2 x-intercepts for the parabola—they are at $x = 7$ and $x = 3$. We can also find the vertex of the parabola by completing the square.

$$
\begin{aligned}
&x^2 - 10x + 21 \\
&= (x^2 - 10x + 25) - 25 + 21 \\
&= (x - 5)^2 - 4
\end{aligned}
$$

So we see that the vertex is positioned at (5, −4). Now we can plot this parabola accurately.
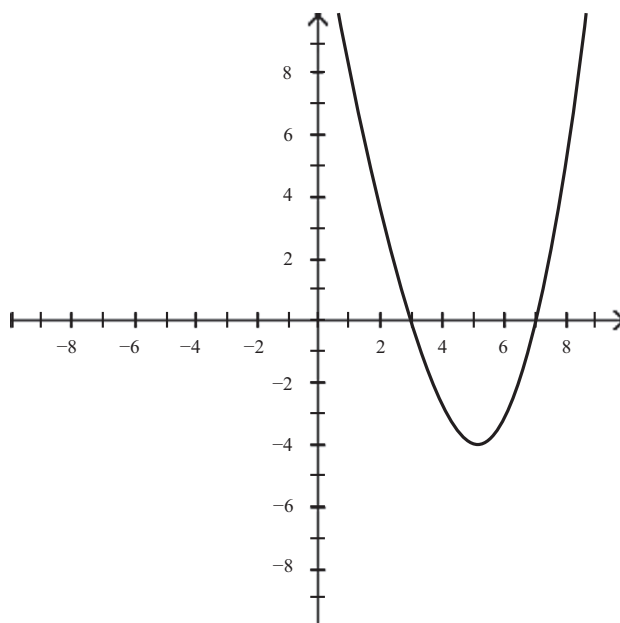
Note that this boundary parabola is drawn with a solid line, not a dashed line, because the original inequality was not a strict "greater than" inequality.

Next, we want all the points in the plane for which the original inequality holds. We shade in the solution set that is the set of all points above the parabola.
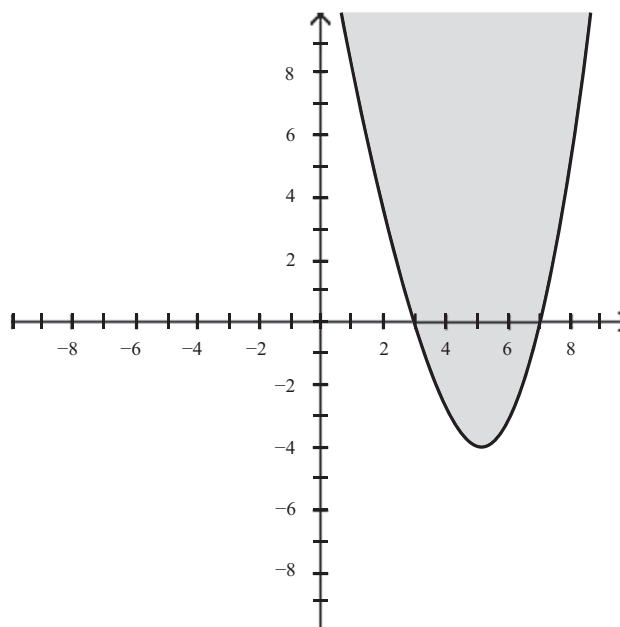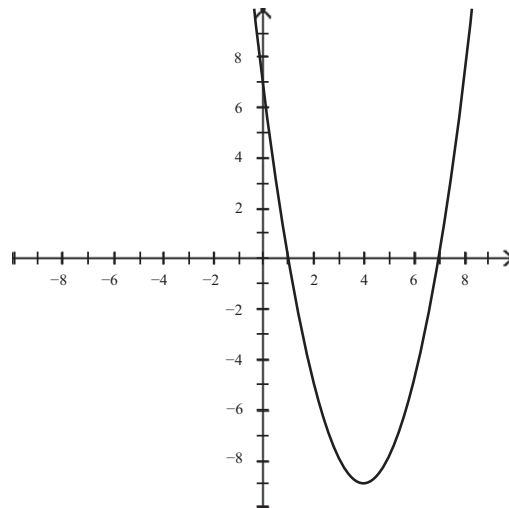
Example 2

Solve the system of inequalities $y \geq x^2 - 8x + 7$ and $y \leq -x^2 + 6x$.

We start by looking at one inequality at a time. First, the equation of the parabola that corresponds to $y \geq x^2 - 8x + 7$ can be rewritten as $y = (x-7)(x-1)$.

So it is a parabola with 2 x-intercepts, one at $x = 7$ and the other at $x = 1$. It can be sketched this way.

The portion of the plane above the parabola is the solution set of the inequality, so we shade it here.
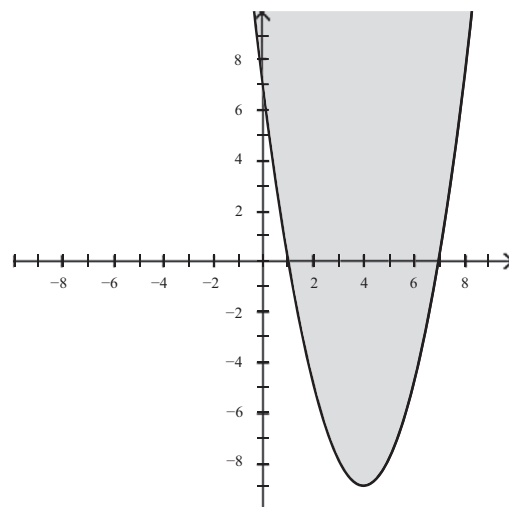
Next, we sketch the graph of the other boundary parabola, the one that corresponds to the equation $y = -x^2 + 6x$. This parabola is an upside-down U shape. The equation factors nicely as $y = x^3 - x + 2$. That means the 2 x-intercepts occur when $x = 0$ and $x = 6$. So this parabola goes through (0, 0) and (6, 0). We can find the vertex of this parabola by completing the square.

$$y = -x^2 + 6x$$
$$y = -(x^2 - 6x)$$
$$y = -(x^2 - 6x + 9) + 9$$
$$y = -(x - 3)^2 + 9$$

So the vertex is at (3, 9). Note that we have to add 9 outside the parentheses in the third equation above. This is because when we add 9 within the parentheses, it actually has the effect of subtracting 9 due to the minus sign in front of the parentheses. To balance everything out, we have to add 9 to the end of the equation.
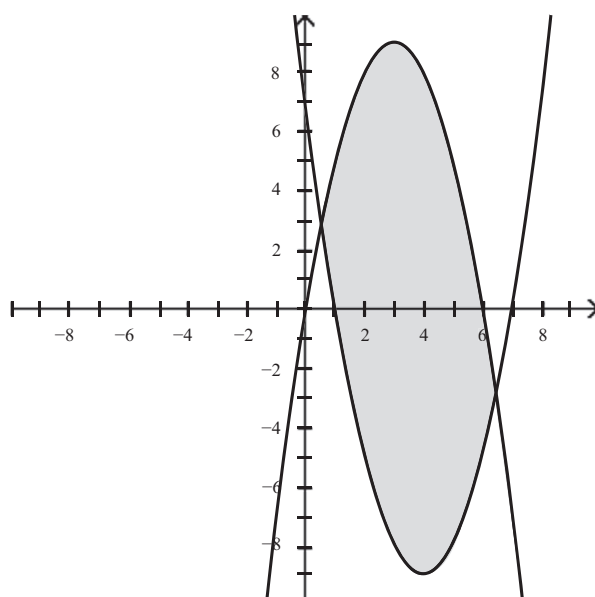
We now sketch this second parabola on the same set of axes.
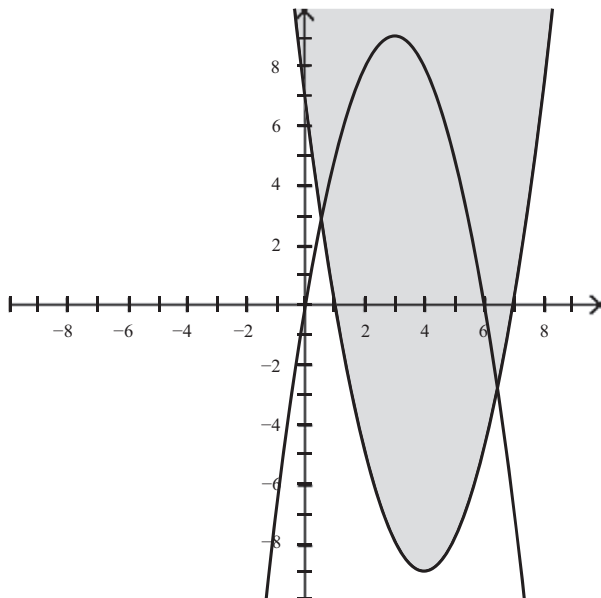
And since this second inequality is a "less than or equal to" inequality, we need to shade in the portion of the *xy*-plane that is below this parabola, so we do that here.
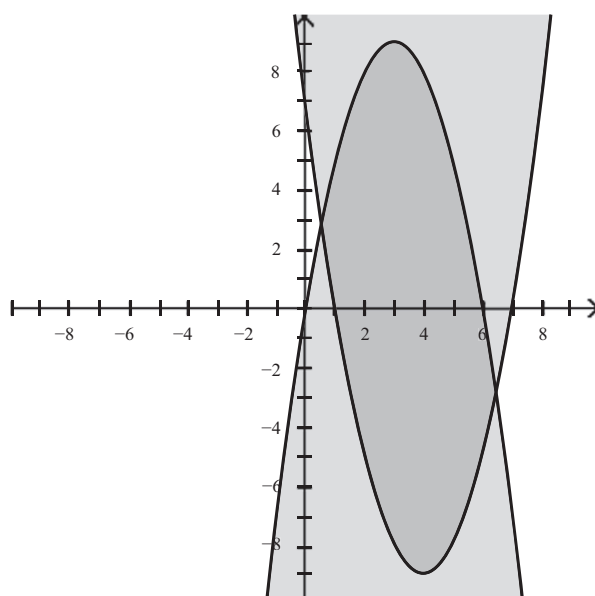

The solution of the system of inequalities is the intersection of these 2 shaded regions.

Example 3

Solve the system of inequalities $y < x^2 + 5x + 6$ and $y < -(x^2 - 10x + 21)$.

We start by drawing the boundary parabola that corresponds to the first inequality. This is given by the equation $y = x^2 + 5x + 6$, which can also be written as $y = (x + 2)(x + 3)$. So this parabola has x-intercepts at $(-2, 0)$ and $(-3, 0)$. We draw it with dashed lines since this is a strict inequality.
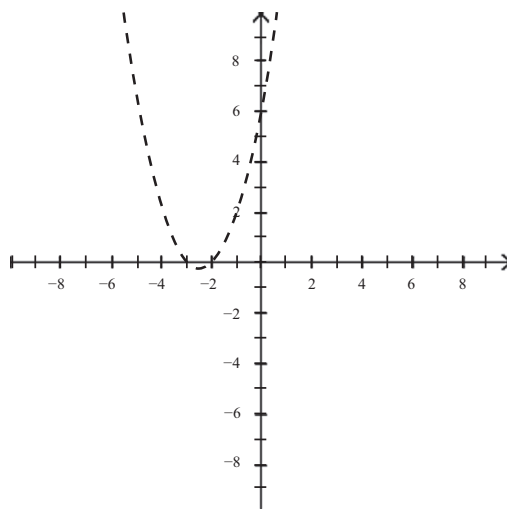
Then the solution of this inequality is the set of all points below this parabola, so the solution set looks like this.
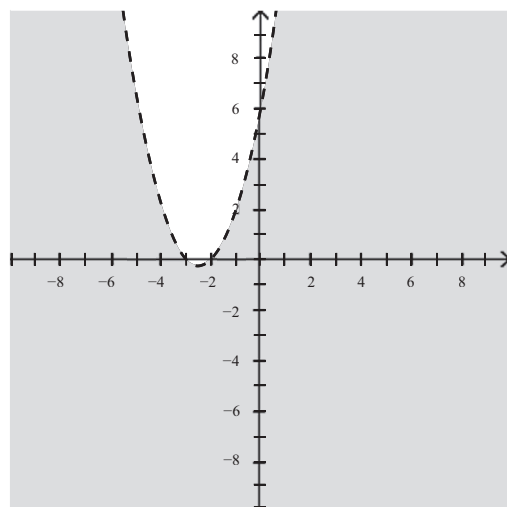
We next draw the second parabola, which is given by the equation $y = -(x^2 - 10x + 21)$ or $y = -(x-7)(x-3)$. The solution set for this second inequality is then the set of points below (or "inside") this second parabola. So we shade in this region as well.
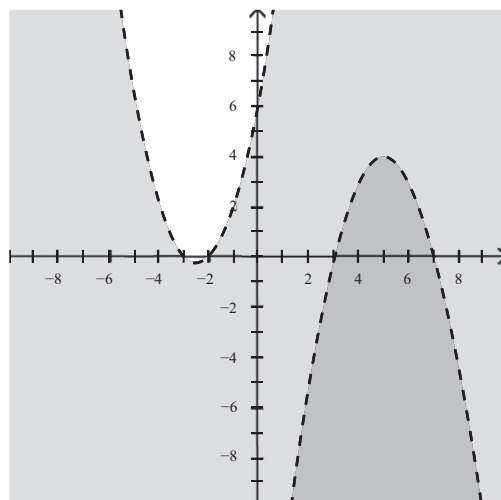
The solution set for the system of inequalities is given by the intersection, below.
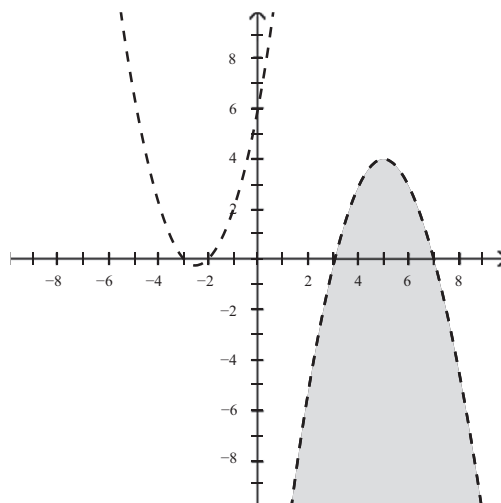
Common Errors

- Forgetting to draw certain boundary parabolas with dashed lines rather than solid lines. Be careful to pay attention to the original inequalities.

Study Tips

- Use graph or grid paper, and draw the parabolas carefully!

- Remember to look for x-intercepts (if they are easy to find) as well as the vertex (which can be found using completing the square). These points will help you draw the parabolas accurately.

Problems

Solve the following quadratic inequalities.

1. $y \geq -x^2 + 1$

2. $y > 5x^2$

3. $y \leq x^2 - 11x + 30$

4. $y < x^2 - 4x + 6$

5. $y > x^2 - 13x + 40$

Solve the following systems of quadratic inequalities.

6. $y \geq x^2 - 1$
   $y < 1 - x^2$

7. $y \geq x^2 + 3$
   $y \geq x^2 - 2x + 1$

8. $y \leq (x + 2)^2 + 4$
   $y > -x^2$

9. $y > x^2 - 4x + 3$
   $y < -x^2 + 8x - 7$

10. $y \geq x^2$
    $y > (x - 3)^2$

Conic Sections—Parabolas and Hyperbolas
Lesson 17

Topics in This Lesson

- General history of conic sections.

- Geometric definition of parabolas and hyperbolas.

- Equations related to parabolas and hyperbolas.

Summary

In this lesson, we discuss a brief history of the conic sections as they were originally studied by Apollonius around 200 B.C. We then focus our attention on 2 of the 4 conic sections—parabolas and hyperbolas. We briefly touch on their geometric definitions and then move to more algebraic topics, like the equations related to parabolas and hyperbolas and how these connect to their graphs.

Definitions and Formulas

directrix: A special line used in the definition of a parabola.

focus: A special point used in the definition of parabolas and other conic sections.

hyperbola: The set of all points in the plane such that the differences of the distances from each of the points on the hyperbola to the 2 foci is a constant amount.

parabola: The set of all points in the plane that are the same distance from the directrix as they are from the focus.
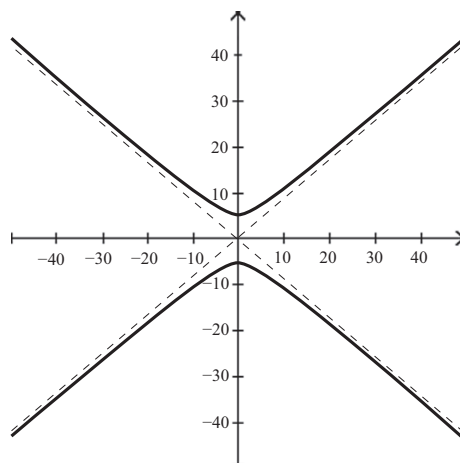
The standard form of the **equation of a hyperbola** looks like $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ or $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$, where a and b are some real numbers. In the case of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, the vertices are at the points $(a, 0)$ and $(-a, 0)$ and the asymptotes are given by $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$. In the case of $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$, the vertices are at the points $(0, a)$ and $(0, -a)$ and the asymptotes are given by $y = \dfrac{a}{b}x$ and $y = -\dfrac{a}{b}x$.

Examples

Example 1

Graph the hyperbola given by the equation $\dfrac{y^2}{25} - \dfrac{x^2}{36} = 1$.

Notice that this equation is already in standard form. In this case, the branches of the hyperbola will open along the y-axis (that is, they will open up and down). The vertices of the hyperbola are located at $(0, 5)$ and $(0, -5)$. We also know that the equations of the asymptotes are $y = (5/6)x$ and $y = (-5/6)x$, since $\sqrt{25} = 5$ and $\sqrt{36} = 6$. Therefore, the hyperbola looks like this.
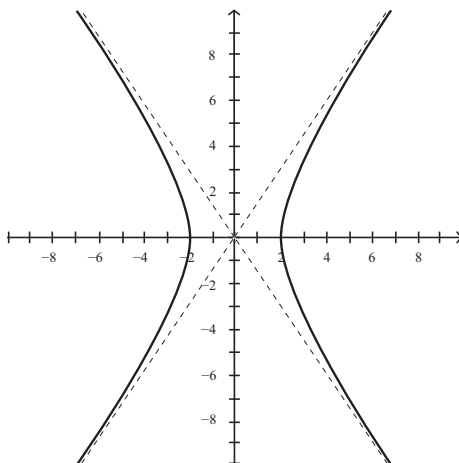

Example 2

Draw the graph of the equation $27x^2 - 12y^2 = 108$.

To get the equation into standard form, we divide both sides by 108. That gives us the following.

$$27x^2 - 12y^2 = 108$$
$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$

In this case, the branches of the hyperbola will open along the x-axis. Also, the vertices will occur at $(2, 0)$ and $(-2, 0)$, since $\sqrt{4} = 2$. The asymptotes will occur at $y = (3/2)x$ and $y = (-3/2)x$.

Common Errors

- Confusing the directions in which the branches of the hyperbola open (between the x-axis and the y-axis).

- Forgetting to be careful when calculating the equations of the asymptotes.

Study Tips

- Use graph paper when drawing these graphs.

- Use the asymptotes to obtain accurate drawings of the hyperbolas.

Problems

State whether each equation corresponds to a parabola or a hyperbola.

1. $x^2 + 5y = 7x$

2. $3x^2 - 4y^2 = 60$

3. $2y^2 = 4x^2 + 100$

State whether the following hyperbolas will open along the x-axis or along the y-axis. Also determine the vertices of each hyperbola.

4. $\dfrac{x^2}{16} - \dfrac{y^2}{49} = 1$

5. $\dfrac{y^2}{9} - \dfrac{x^2}{64} = 1$

6. $\dfrac{y^2}{81} - \dfrac{x^2}{25} = 1$

Sketch the graph of each hyperbola.

7. $\dfrac{x^2}{121} - \dfrac{y^2}{81} = 1$

8. $\dfrac{x^2}{16} - \dfrac{y^2}{49} = 1$

9. $y^2 - \dfrac{x^2}{16} = 1$

10. $\dfrac{y^2}{64} - \dfrac{x^2}{9} = 1$

Conic Sections—Circles and Ellipses
Lesson 18

Topics in This Lesson

- Continued discussion of conic sections.

- Ellipses.

- Circles.

Summary

We continue our discussion of conic sections (which began in Lesson 17) by talking about the other 2 types of conic sections—ellipses and circles. We talk about these geometrically as well as algebraically.

Definitions and Formulas

center of the circle: The point from which all the points on a circle are equidistant (this is basically the point in the center of the circle).

circle: The set of all points that are a given distance from a special point (called the center of the circle).

eccentricity of the ellipse: The length of the minor axis divided by the length of the major axis.

ellipse: The set of all points in the plane such that the sum of the distances from each of the points on the ellipse to the 2 foci is a constant amount.

major axis of the ellipse: The longer line segment in the axis of the ellipse.

minor axis of the ellipse: The shorter line segment in the axis of the ellipse.

vertices of the ellipse: The endpoints of the major axis of the ellipse.

The standard form of the **equation of an ellipse** is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where a and b are some nonzero numbers and $a > b$. Note that a is half the length of the major axis and b is half the length of the minor axis.

The standard form for the **equation of a circle** centered on the origin is given by $x^2 + y^2 = r^2$, where r is the radius of the circle.

Examples

Example 1

Determine the graph of the equation $\dfrac{x^2}{49} + \dfrac{y^2}{81} = 1$.

The equation is in standard form for an ellipse, and the larger denominator is under the y^2 term, so this graph will be stretched along the y-axis. The vertices of the ellipse will be at (0, 9) and (0, −9), since $\sqrt{81} = 9$. These are also the endpoints of the major axis. The endpoints of the minor axis are (7, 0) and (−7, 0). We plot these 4 points and connect the dots to complete the sketch of the graph.
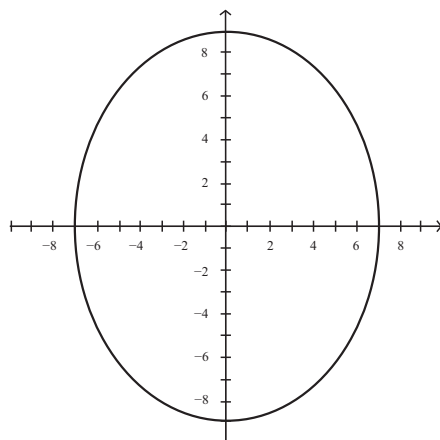
Example 2

Sketch the graph of the equation $4x^2 = 36 - 4y^2$.

We begin by getting the equation into standard form. We add $4y^2$ to the left-hand side, which gives us $4x^2 + 4y^2 = 36$. Next, we divide both sides of the equation by 4 to obtain $x^2 + y^2 = 9$. This is the equation of a circle. The graph is a circle centered at (0, 0) of radius 3, since $\sqrt{9} = 3$.
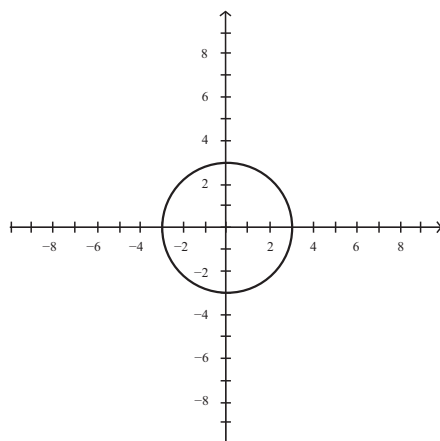
Example 3

Find the eccentricity of the ellipse given by the equation $\dfrac{x^2}{49} + \dfrac{y^2}{81} = 1.$

Since $\sqrt{49} = 7$ and $\sqrt{81} = 9$, we know the eccentricity of this ellipse is $\dfrac{7}{9}$.

Common Errors

- Switching the orientation of an ellipse (whether it should be elongated along the x-axis or the y-axis).

Study Tips

- Graph paper is always helpful when drawing these graphs.

Problems

Determine whether the following equations correspond to ellipses or circles.

1. $\dfrac{x^2}{100} + \dfrac{y^2}{121} = 1$

2. $\dfrac{x^2}{100} + \dfrac{y^2}{100} = 1$

3. $6x^2 + 7y^2 = 42$

Find the eccentricity of each ellipse given by these equations.

4. $\dfrac{x^2}{9} + \dfrac{y^2}{169} = 1$

5. $\dfrac{x^2}{81} + \dfrac{y^2}{64} = 1$

6. $x^2 + \dfrac{y^2}{25} = 1$

Sketch the graph of each of these equations.

7. $\dfrac{x^2}{9} + \dfrac{y^2}{225} = 1$

8. $\dfrac{x^2}{225} + \dfrac{y^2}{9} = 1$

9. $\dfrac{x^2}{25} + \dfrac{y^2}{25} = 1$

10. $25x^2 + 16y^2 = 400$

An Introduction to Polynomials
Lesson 19

Topics in This Lesson

- The definition of a polynomial.

- The degree of a polynomial.

- Classifying polynomials by degree.

- Classifying polynomials by the number of terms.

- Evaluating a polynomial at a value of the variable.

Summary

In this lesson, we define polynomials and begin the process of classifying them (by degree and by the number of terms present). We also discuss how to evaluate a polynomial at a value of the variable.

Definitions and Formulas

degree of a polynomial: The largest power of the variable in the polynomial.

leading coefficient of a polynomial: The coefficient (or number) in front of the term that contains the largest power of x in a polynomial.

polynomial: An expression of the form $a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$, where each of the a's are real numbers (which could be positive, negative, or zero) and the powers on all the x's are positive integers.

standard form of a polynomial: The way of writing a polynomial such that the powers on the variables decrease as you read the polynomial from left to right.

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monomial: A polynomial containing only 1 term.

binomial: A polynomial containing exactly 2 terms.

trinomial: A polynomial containing exactly 3 terms.

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linear polynomial: A polynomial of degree 1.

quadratic polynomial: A polynomial of degree 2.

cubic polynomial: A polynomial of degree 3.

quartic polynomial: A polynomial of degree 4.

quintic polynomial: A polynomial of degree 5.

Examples

Example 1

Classify the polynomial $5x^3 - 1$ by degree and by number of terms.

This polynomial has degree 3, so it is a cubic polynomial. It has 2 terms, so it is a binomial. So you can call this a cubic binomial.

Example 2

Classify the polynomial $x^4 - \dfrac{1}{3}x + \dfrac{2}{9}$ by degree and by number of terms.

This polynomial has degree 4, so it is a quartic polynomial. It has 3 terms, so it is a trinomial. It is a quartic trinomial.

Example 3

Determine $f(0)$, $f(1)$, $f(2)$, and $f(-1)$ for the function $f(x) = 2x^4 - 3x + 5$.

$$f(x) = 2x^4 - 3x + 5$$

$$f(0) = 2(0)^4 - 3(0) + 5$$
$$f(0) = 2(0) - 0 + 5$$
$$f(0) = 0 - 0 + 5$$
$$f(0) = 5$$

$$f(1) = 2(1)^4 - 3(1) + 5$$
$$f(1) = 2(1) - 3(1) + 5$$
$$f(1) = 2 - 3 + 5$$
$$f(1) = 4$$

$$f(2) = 2(2)^4 - 3(2) + 5$$
$$f(2) = 2(16) - 3(2) + 5$$
$$f(2) = 32 - 6 + 5$$
$$f(2) = 31$$

$$f(-1) = 2(-1)^4 - 3(-1) + 5$$
$$f(-1) = 2(1) - 3(-1) + 5$$
$$f(-1) = 2 + 3 + 5$$
$$f(-1) = 10$$

Example 4

Determine whether the following are polynomials. If so, then state the degree of the polynomial.

    a. $7 - 3x^2 + x^5$

    b. $x^2 + x^{-1}$

    c. $\dfrac{x+2}{x-4}$

Solutions

    a.   This is a polynomial of degree 5. (Note that the largest power on x is 5.)

    b.   This is not a polynomial, because it contains a negative exponent.

    c.   This is also not a polynomial; it is the ratio of 2 polynomials.

Common Errors

- Misidentifying the degree of a polynomial when it has not been written in standard form. Be careful of this.

- Confusing the words for the degree of the polynomial (linear, quadratic, cubic, quartic, quintic) with the words for the number of terms in a polynomial (monomial, binomial, trinomial).

Study Tips

- When evaluating a polynomial at a number, remember to use order of operations carefully.

- Remember that raising a negative number to an even power will "cancel" the negative sign.

Problems

Determine whether the following are polynomials.

1. $\sqrt{x^2 + 1}$

2. $(x - 2)^2 + 4x - 1$

3. $\sqrt{2}x^3 - 100x + \pi$

4. $x^5 + x^3 - x^{-3} - x^{-5}$

Classify each of these polynomials by degree and by number of terms.

5. $12x^3 - x + 2$

6. $-4x + 9x^3 - x^5$

7. $x^{10} - 1$

Evaluate each of the following polynomials at $x = 0$ and $x = 1$.

8. $f(x) = 6x^2 + 12x - 3$

9. $f(x) = -x^4 + 2x + 13$

10. $f(x) = x^5 + x^4 + x^3 + x^2 + x$

Graphing Polynomial Functions
Lesson 20

Topics in This Lesson

- Graphs of polynomials.

- The behavior of the "ends" of the graph of a polynomial.

- The number of x-intercepts of a graph of a polynomial.

Summary

We've spent a good deal of time talking about the graphs of linear functions (lines) and quadratic polynomials (parabolas). In this lesson, we talk about the graphs of general polynomial functions. Although we can't generalize about every aspect of such graphs, we can say something about the smoothness of such graphs and about the behavior of such graphs at their "ends" and x-intercepts.
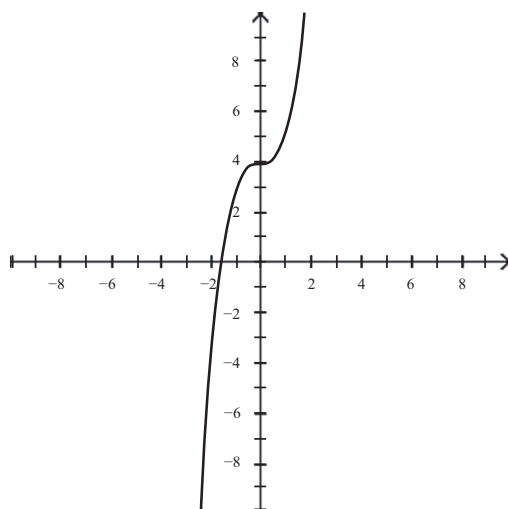
Examples

Example 1
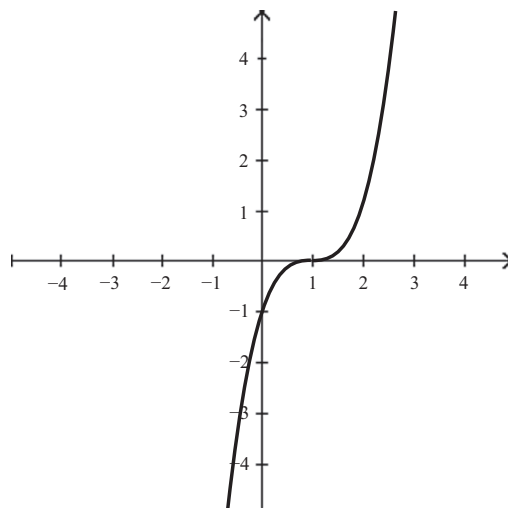
Given that the graph of $f(x) = x^3$ is shown below,
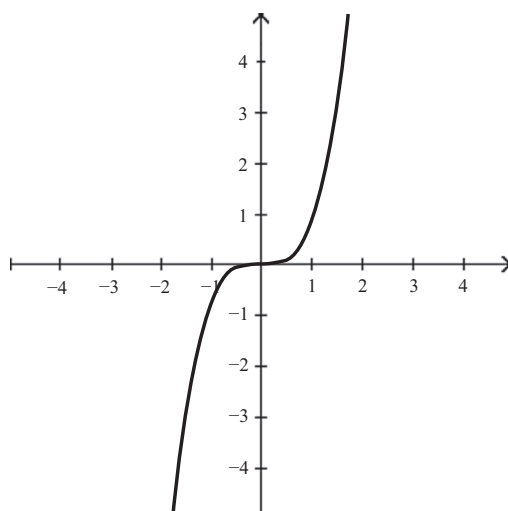
sketch the graph of the cubic functions $g(x) = x^3 + 4$ and $h(x) = (x-1)^3$.

We know that the graph of $g(x)$ must look like the graph of $f(x)$, just translated upward by 4 units. So the graph of $g(x)$ is the following.

The graph of $h(x)$ is the same as the graph of $f(x)$ except it is shifted to the right by exactly 1 unit. So the graph of $h(x)$ is below.

Example 2

Sketch the graph of $f(x) = x^4 - 4x^2$.

We begin by factoring the function.

$$f(x) = x^4 - 4x^2$$
$$f(x) = x^2(x^2 - 4)$$
$$f(x) = x^2(x - 2)(x + 2)$$

This tells us that the *x*-intercepts of the graph of this function are located at $x = 0$, $x = -2$, and $x = 2$.

Next, let's talk about end behavior. Since this function has degree 4, and that degree is even, we know that both ends of the graph will go in the same direction (either both will go upward or both will go downward). In this case, since the coefficient in front of the x^4 term is positive (it is +1), we know that both ends will go up.

Lastly, we should plot several points (maybe by plugging in $x = 1, 2, 3, 4, 5$ and $x = -1, -2, -3, -4, -5$). After plotting those points, we would have a graph that looks like the following.
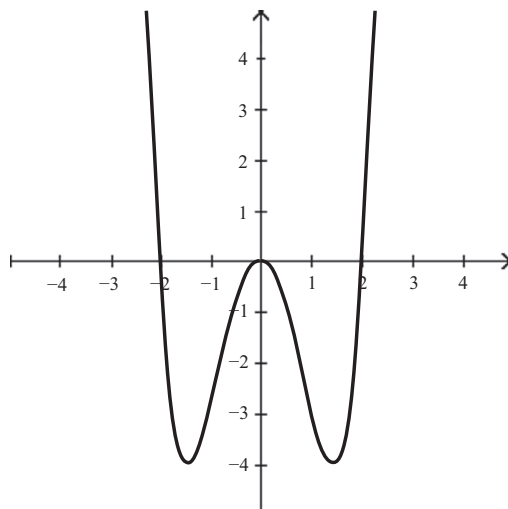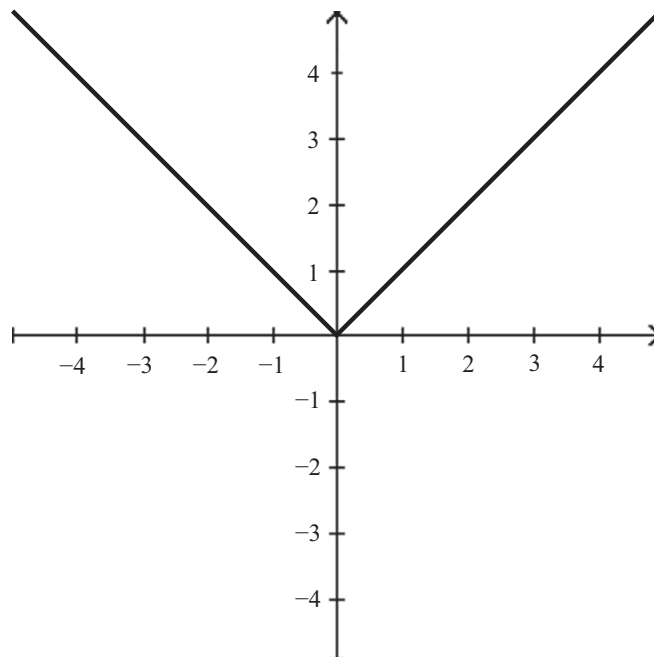

Study Tips

- Factoring or rewriting a polynomial in a different way can often lead to helpful insights about how to draw the graph of the polynomial.

- When you are not sure what the shape of a graph is, plot several points and simply connect the dots.
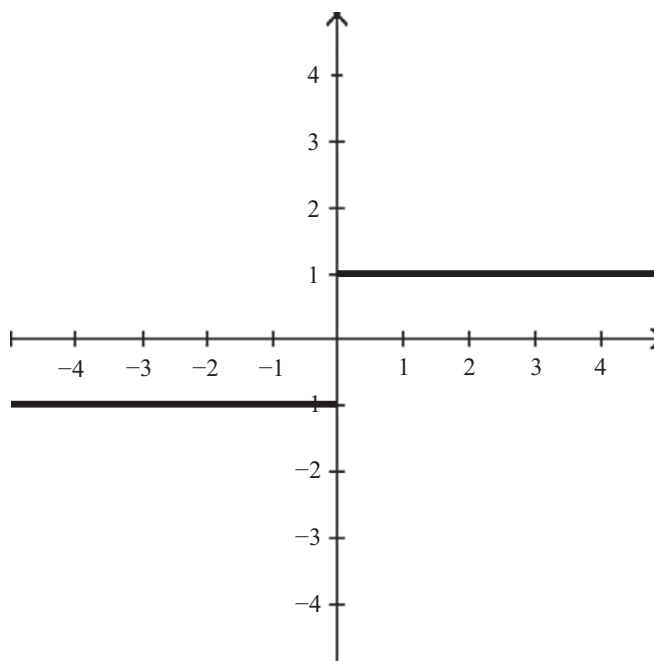
Problems

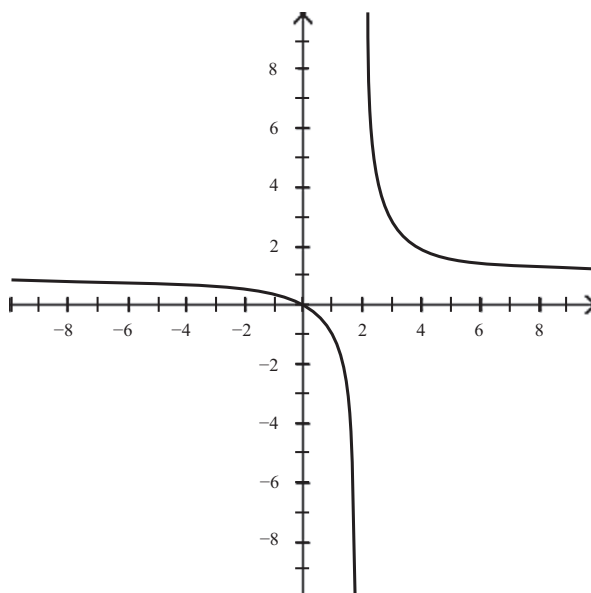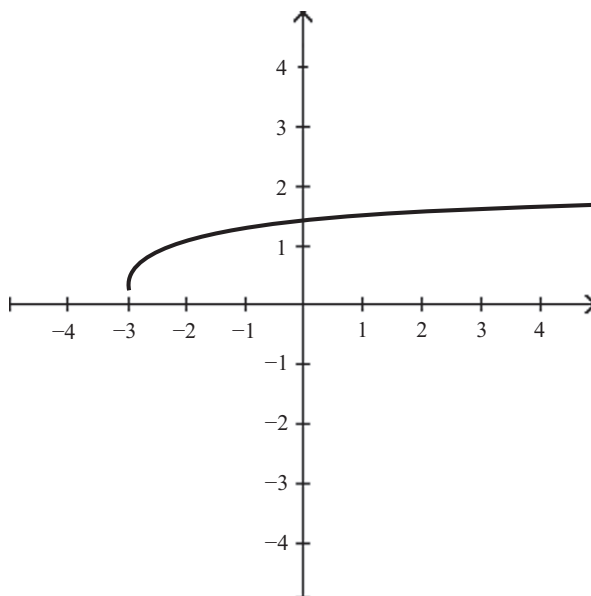Determine which of the following are graphs of polynomials and which are not.

1.

2.

3.

4.

Describe the end behavior of the graphs of each of the following polynomials.

5. $3x^6 - 2x + 7$

6. $-4x^5 + 3x^4 + 2x^2$

7. $-10x^8 + 20$

Sketch the graph of each of the following polynomials.

8. $f(x) = -x(x-2)(x-4)(x-6)$

9. $f(x) = x^3 - x$

10. $f(x) = (x^2 - 1)^2$

Combining Polynomials
Lesson 21

Topics in This Lesson

- Combining polynomials using the operations of addition, subtraction, multiplication, and division.

- Composing 2 polynomials.

Summary

We discuss combining 2 polynomials by using the usual operations of addition, subtraction, and multiplication (all of which produce new polynomials). We also speak about dividing one polynomial by another, which most often does not produce a new polynomial. Lastly, we talk about how to compose one function with another.

Definitions and Formulas

composing: The process of replacing the variables in one function with a different function.

Examples

Example 1

Expand $(x^2 - 7)(x^3 + 9x^2 - 5x)$.

The key is to have every term in the first set of parentheses multiplied with every term in the other set of parentheses.

$$\begin{aligned}
&(x^2 - 7)(x^3 + 9x^2 - 5x) \\
&= x^2(x^3) + x^2(9x^2) + x^2(-5x) - 7(x^3) - 7(9x^2) - 7(-5x) \\
&= x^5 + 9x^4 - 5x^3 - 7x^3 - 63x^2 + 35x \\
&= x^5 + 9x^4 - 12x^3 - 63x^2 + 35x
\end{aligned}$$

Example 2

Determine $f(g(x))$ and $g(f(x))$ if $f(x) = x^3 + 3x - 1$ and $g(x) = x^2 - 2$.

$$f(g(x))$$
$$= f(x^2 - 2)$$
$$= (x^2 - 2)^3 + 3(x^2 - 2) - 1$$
$$= x^6 - 6x^4 + 12x^2 - 8 + 3x^2 - 6 - 1$$
$$= x^6 - 6x^4 + 15x^2 - 15$$

$$g(f(x))$$
$$= g(x^3 + 3x - 1)$$
$$= (x^3 + 3x - 1)^2 - 2$$
$$= x^6 + 3x^4 - x^3 + 3x^4 + 9x^2 - 3x - x^3 - 3x + 1 - 2$$
$$= x^6 + 6x^4 - 2x^3 + 9x^2 - 6x - 1$$

Example 3

Find $(x^4 - 7x^3 + 9x^2 - 5x + 3) - (2x^4 - 3x^3 + 8x^2 + 4x - 7)$.

$$(x^4 - 7x^3 + 9x^2 - 5x + 3) - (2x^4 - 3x^3 + 8x^2 + 4x - 7)$$
$$= x^4 - 7x^3 + 9x^2 - 5x + 3 - 2x^4 + 3x^3 - 8x^2 - 4x + 7$$
$$= -x^4 - 4x^3 + x^2 - 9x + 10$$

Common Errors

- Forgetting to distribute all minus signs when subtracting one polynomial from another or when multiplying 2 polynomials.

- Forgetting to multiply all terms in one polynomial by all terms in the second polynomial when combining by multiplying.

- Making mistakes when performing compositions of functions—it is easy to make errors when inserting one function into the other.

Study Tips

- Take your time when simplifying these problems; it is very easy to make mistakes with the arithmetic.

Problems

Simplify the following.

1. $(x^5 - 2x^3 + 4x) + (x^3 + 5x^2 - 6x + 7)$

2. $(x^6 - 2x^4 + 3x^2 + 10) - (x^3 + 5x^2 - 6x + 7)$

3. $(x^4 - 1)(x^2 + 2x + 1)$

4. $(x^2 + 3x + 2)(2x^2 - 5x - 6)$

5. $\dfrac{x^2 - 7x + 12}{x - 3}$

6. $\dfrac{x^3 + x^2 - x - 10}{x - 2}$

Compute $f(g(x))$ and $g(f(x))$ for the following pairs of functions.

7. $f(x) = 4x + 3$ and $g(x) = 5x - 7$

8. $f(x) = x^2 + 1$ and $g(x) = x^2 - 7$

9. $f(x) = x^2 + 3x$ and $g(x) = 5x - 1$

10. $f(x) = x^3 + 3x$ and $g(x) = x^2$

Topics in This Lesson

- Solving specific polynomial equations using factoring.

Summary

In this lesson, we consider some specific polynomial equations (of degree larger than 2) that can be solved by factoring.

Definitions and Formulas

factoring using differences of 2 cubes:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Examples

Example 1

Solve the equation $2x^4 - 32x^2 = 0$.

We begin by factoring out the greatest common factor of the left-hand side of the equation, $2x^2$.

$$2x^4 - 32x^2 = 0$$
$$2x^2(x^2 - 16) = 0$$
$$2x^2(x - 4)(x + 4) = 0$$

So there are 3 solutions here: $x = 0$, $x = 4$, and $x = -4$.

Example 2

Find all real number solutions for the equation $40x^3 + 5 = 0$.

We start by factoring out a common term.

$$40x^3 + 5 = 0$$
$$5(8x^3 + 1) = 0$$

The factor 5 is not going to provide any solutions for this equation. So we can simply divide both sides by 5 and then rewrite the remaining equation in such a way that it fits the "difference of 2 cubes" formula.

$$8x^3 + 1 = 0$$
$$(2x)^3 + 1 = 0$$
$$(2x+1)((2x)^2 - 2x + 1) = 0$$
$$(2x+1)(4x^2 - 2x + 1) = 0$$

The term $2x+1$ gives the solution $x = -1/2$. What about the other factor in the equation, $4x^2 - 2x + 1$? The discriminant of this quadratic expression is negative, so it will not provide any additional real number solutions. So there is only one real number solution to the original equation, $x = -1/2$.

Example 3

Find all real solutions of $x^4 + x^2 - 90 = 0$.

We begin by rewriting this equation in a different way.
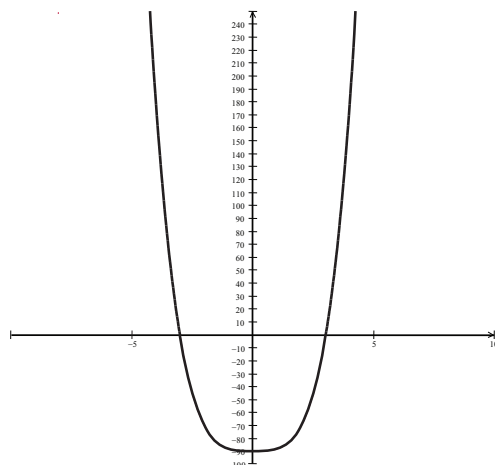
$$x^4 + x^2 - 90 = 0$$
$$(x^2)^2 + x^2 - 90 = 0$$

This now factors as follows.

$$(x^2 + 10)(x^2 - 9) = 0$$
$$(x^2 + 10)(x - 3)(x + 3) = 0$$

So there are 2 real number solutions: $x = 3$ and $x = -3$. The factor $x^2 + 10$ will only provide complex number solutions. We can see that there are only 2 real number solutions by sketching the graph of $f(x) = x^4 + x^2 - 90$ and noting that there are only 2 x-intercepts.

Common Errors

- Falling into the trap of saying that a polynomial equation such as $x^2 + a^2 = 0$ has the real number solutions $x = a$ and $x = -a$. Remember: The solutions of $x^2 + a^2 = 0$ are complex numbers, $x = ai$ and $x = -ai$.

Study Tips

- Remember to factor out the greatest common factor before attempting to identify how to finish the factorization of the polynomial. By clearing out that common factor, you make the degree of the remaining polynomial smaller, which can be very helpful in finishing the problem.

Problems

Find all the real number solutions of the following using factoring.

1. $x^6 - 12x^5 + 27x^4 = 0$

2. $3x^4 - 27x^2 = 0$

3. $2x^5 + 40x^4 + 81x^3 - 5x^2 = x^5 + 21x^4 - 3x^3 - 5x^2$

4. $x^3 - 64 = 0$

5. $4x^5 - 32x^2 = 0$

6. $5x^6 + 320x^3 = 0$

7. $16x^4 + 250x = 0$

8. $x^4 - 11x^2 + 28 = 0$

9. $3x^4 + 20x^2 + 50 = 2x^4 + 5x^2 - 4$

10. $x^6 - 64 = 0$

Rational Roots of Polynomial Equations
Lesson 23

Topics in This Lesson

- The rational roots theorem.

- The factor theorem.

Summary

We continue our study of finding roots of polynomial equations (of degree greater than 2) by studying 2 very important facts in algebra—the rational roots theorem and the factor theorem.

Definitions and Formulas

factor theorem: The expression $x - a$ is a linear factor of a polynomial function if and only if the value a is a zero of the polynomial function.

rational roots theorem: The only possible rational roots of the polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$ are of the form p/q, where p/q is already in reduced form, p is a factor of a_0, and q is a factor of a_n.

Examples

Example 1

Find the rational roots of the equation $x^3 - 3x^2 + x - 3 = 0$.

According to the rational roots theorem, the only possible rational roots occur at fractions of the form p/q, where p is a divisor of 3 and q is a divisor of 1. Since the only divisors of 3 are 3 and 1, and the only divisor of 1 is 1, the only possible rational roots are at 1/1, −1/1, 3/1, and −3/1 (we need to remember to look at the negative values as well as the positive values). So there are only 4 choices: 1, −1, 3, and −3. Now we simply plug each one in for x and see if the equation is true.

$$x^3 - 3x^2 + x - 3 = 0$$
$$(1)^3 - 3(1)^2 + 1 - 3 = 0$$
$$1 - 3 + 1 - 3 = 0$$
$$-4 = 0$$

This is a false statement, so 1 is not a rational root of the equation.

$$x^3 - 3x^2 + x - 3 = 0$$
$$(-1)^3 - 3(-1)^2 + (-1) - 3 = 0$$
$$-1 - 3 - 1 - 3 = 0$$
$$-8 = 0$$

This is also a false statement, so −1 is not a rational root of the equation.

$$x^3 - 3x^2 + x - 3 = 0$$
$$(3)^3 - 3(3)^2 + 3 - 3 = 0$$
$$27 - 27 + 3 - 3 = 0$$
$$0 = 0$$

This is a true statement, so 3 is a rational root of the equation.

$$x^3 - 3x^2 + x - 3 = 0$$
$$(-3)^3 - 3(-3)^2 + (-3) - 3 = 0$$
$$-27 - 27 - 3 - 3 = 0$$
$$-60 = 0$$

This is false, so −3 is not a rational root of the equation. Thus, we can conclude that there is only one rational root of this equation, $x = 3$.

Example 2

Use the rational roots theorem and the factor theorem to find all the real roots (rational and not rational) for the polynomial equation $x^3 - 6x^2 + 6x + 4 = 0$.

We begin by finding rational roots of this equation. From the rational roots theorem, we know that the only choices for rational roots are of the forms 1/1, −1/1, 2/1, −2/1, 4/1, and −4/1. So there are 6 choices that can be more simply written as 1, −1, 2, −2, 4, or −4.

Let's see if any of these is a rational root.

First we check $x = 1$.

$$x^3 - 6x^2 + 6x + 4 = 0$$
$$(1)^3 - 6(1)^2 + 6(1) + 4 = 0$$
$$1 - 6 + 6 + 4 = 0$$
$$5 = 0$$

This is a false statement.

Let's try $x = -1$.

$$x^3 - 6x^2 + 6x + 4 = 0$$
$$(-1)^3 - 6(-1)^2 + 6(-1) + 4 = 0$$
$$-1 - 6 - 6 + 4 = 0$$
$$-9 = 0$$

This is also a false statement.

Now $x = 2$.

$$x^3 - 6x^2 + 6x + 4 = 0$$
$$(2)^3 - 6(2)^2 + 6(2) + 4 = 0$$
$$8 - 24 + 12 + 4 = 0$$
$$0 = 0$$

This is a true statement, so we know $x = 2$ is a root of the original equation. Thanks to the factor theorem, this means $x - 2$ is a factor of the original left-side polynomial. We can then factor this out of the left-hand side, leaving us with the following.

$$x^3 - 6x^2 + 6x + 4 = 0$$
$$(x - 2)(x^2 - 4x - 2) = 0$$

We now see that we can find the roots of the other part of $x^2 - 4x - 2 = 0$ using a different tool, such as the quadratic formula. We do so here.

$$x^2 - 4x - 2 = 0$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-2)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{24}}{2}$$

$$x = \frac{4 \pm 2\sqrt{6}}{2}$$

$$x = 2 \pm \sqrt{6}$$

Therefore, we see that we now know all the real roots. There is one rational root, which is 2. The other 2 roots are $2 + \sqrt{6}$ and $2 - \sqrt{6}$.

Common Errors

- Forgetting that a root of $+a$ means that there is a factor of $x - a$ in the original polynomial, not $x + a$. Remember that the signs must be switched.

Study Tips

- Remember to look for both positive and negative rational roots.

- Once you find a root, say a, go back to the original equation and factor out $x - a$ (this is what the factor theorem tells us). Then you will have an equation of smaller degree with which to work. There may be a variety of tools you can then use to solve this smaller equation.

Problems

Given the rational roots theorem, state all possible rational roots for each equation.

1. $x^5 - 3x^4 + 5x^3 - 15x^2 + 4x - 12 = 0$

2. $x^3 - 8x^2 + 22x - 35 = 0$

3. $2x^3 + x^2 - 2x - 1 = 0$

Find all the rational roots of each equation using the rational roots theorem.

4. $x^5 - 3x^4 + 5x^3 - 15x^2 + 4x - 12 = 0$

5. $x^3 - 8x^2 + 22x - 35 = 0$

6. $2x^3 + x^2 - 2x - 1 = 0$

Find all the real roots (rational and not rational) for each equation.

7. $x^4 - 8x^3 + 2x^2 + 8x - 3 = 0$

8. $x^4 + x^3 - 3x^2 - 4x - 4 = 0$

9. $6x^4 + 5x^3 + 7x^2 + 5x + 1 = 0$

10. $2x^4 + 5x^3 + 4x^2 + x = 0$

The Fundamental Theorem of Algebra
Lesson 24

Topics in This Lesson

- Descartes' rule of signs.

- The fundamental theorem of algebra.

Summary

We have spent a number of lessons learning how to find roots of polynomial equations. In this lesson, we finish this topic by sharing 2 additional tools for locating roots or knowing how many you should have. These 2 tools are known as Descartes' rule of signs and the fundamental theorem of algebra.

Definitions and Formulas

by-product of the fundamental theorem of algebra: If you include all roots (rational, real but not rational, and imaginary) and count the roots that appear multiple times, it turns out that an nth-degree polynomial equation has exactly n roots!

Descartes' rule of signs: Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$; let C_p equal the number of sign changes between the coefficients (when they are read left-to-right) of $p(x)$; and let C_n equal the number of sign changes between the coefficients of $p(-x)$. Then the number of positive real zeros of $p(x)$ is C_p, $C_p - 2$, $C_p - 4$, ..., down to either 1 or 0 (depending on whether C_p is odd or even); and the number of negative real zeros of $p(x)$ is C_n, $C_n - 2$, $C_n - 4$, ..., down to either 1 or 0 (depending on whether C_n is odd or even).

fundamental theorem of algebra: If $p(x)$ is a polynomial of degree at least 1, then the equation $p(x) = 0$ has at least one root.

Examples

Example 1

Use Descartes' rule of signs to determine how many positive and negative solutions there can be for the polynomial equation $5x^7 - 8x^6 + 19x^3 + 2x^2 - 14x - 8 = 0$.

Descartes' rule of signs first says to start with the polynomial in standard form (and our left-hand side is already in standard form). Next, we see how many times the sign changes as we read from left to right. The signs of the coefficients go as follows: $+ -, - +, + +, + -, - -$.

There are 3 times where the signs change from either plus to minus or minus to plus. That means that the possible number of positive roots is either 3 (which is the number of sign changes we saw above) or 1. Where did the 1 come from? Once you see the number of sign changes, you start there and keep subtracting down by 2 to see all other possible amounts.

What about the possible number of negative roots? We go back to the original polynomial and plug in –x for all the x's in the polynomial. That would give us the following.

$$5(-x)^7 - 8(-x)^6 + 19(-x)^3 + 2(-x)^2 - 14(-x) - 8$$

That simplifies as

$$-5x^7 - 8x^6 - 19x^3 + 2x^2 + 14x - 8.$$

What are the sign changes? They are $--$, $--$, $-+$, $++$, and $+-$. There are 2 sign changes in this list. This means that the possible number of negative roots is either 2 or 0. (Remember that we need to start with the number of sign changes we see and then keep subtracting 2 until we get down to either 0 or 1.)

Example 2

Determine the roots of the equation

$$x^3 + 5x^2 + 13x + 21 = 0.$$

Thanks to the fundamental theorem of algebra, we know that there have to be 3 roots (because we count how many times each one appears and keep track of both complex and real roots). Thanks to Descartes' rule of signs, we see that there are 0 positive real roots because the left-side polynomial has no sign changes between pairs of consecutive coefficients. Next, consider the left-hand side with x replaced by –x (in order to look for negative real roots).

$$(-x)^3 + 5(-x)^2 + 13(-x) + 21$$
$$= -x^3 + 5x^2 - 13x + 21$$

There are 3 sign changes here, so the number of negative real roots for the original equation must be either 1 or 3. From the rational roots theorem, we know that the potential negative rational roots are −1, −3, −7, and −21. Let's check these.

$$(-1)^3 + 5(-1)^2 + 13(-1) + 21 = 0$$
$$-1 + 5 - 13 + 21 = 0$$
$$12 = 0$$

This is false, so −1 is not a solution of the original equation.

$$(-3)^3 + 5(-3)^2 + 13(-3) + 21 = 0$$
$$-27 + 45 - 39 + 21 = 0$$
$$0 = 0$$

This is true, so $x = -3$ is a solution. This means we can now factor $x + 3$ from the left-hand side of the original equation.

$$x^3 + 5x^2 + 13x + 21 = 0$$
$$(x+3)(x^2 + 2x + 7) = 0$$

Now we just need to consider whether there are any roots of the equation $x^2 + 2x + 7 = 0$. But this can now be determined by the quadratic formula.

$$x = \frac{-2 \pm \sqrt{4 - 28}}{2}$$

$$x = \frac{-2 \pm \sqrt{-24}}{2}$$

$$x = \frac{-2 \pm 2\sqrt{-6}}{2}$$

$$x = -1 \pm \sqrt{-6}$$

$$x = -1 \pm \sqrt{6}i$$

So there are no additional real roots, just these 2 complex roots along with the negative real root, $x = -3$.

Common Errors

- Forgetting that Descartes' rule of signs only speaks to real roots; it does not address complex roots.

- Forgetting that if there is no sign in front of the leading coefficient, then that coefficient is positive.

Study Tips

- Memorize all of the tools we have developed for solving equations. There are many of them, but keeping track of each (what it says and doesn't say) is very important.

Problems

Use Descartes' rule of signs to determine how many possible positive real solutions and negative real solutions there are.

1. $2x^4 + 3x - 50 = 0$

2. $7x^8 - 3x^4 + 5x = 0$

3. $17x^5 + 3x^4 + 8x^3 + 8x^2 - 10x - 14 = 0$

4. $6x^5 - 3x^4 - 8x^3 + 8x^2 + 10x - 14 = 0$

5. $3x^6 - 5x^2 + 4 = 0$

Find all the roots of these equations (real or complex).

6. $x^3 - 11x^2 - 24x - 10 = x + 3$

7. $x^4 = 6x^2 - 8$

8. $x^3 - x^2 + x - 1 = 0$

9. $x^4 = 256$

10. $x^6 + 4x^4 = x^2 + 4$

Roots and Radical Expressions
Lesson 25

Topics in This Lesson

- The idea of the nth root of a number.

- Simplifying a variety of expressions involving n^{th} roots.

- Rationalizing the denominator of a fraction.

Summary

In this lesson, we shift gears to the idea of roots, like square roots, only more general. After defining roots, we then consider a wide variety of simplification examples as well as the idea of rationalizing the denominator.

Definitions and Formulas

n^{th} root: For any real numbers a and b, and any positive integer n, if $a^n = b$, then a is an nth root of b.

Examples

Example 1

Simplify $\sqrt[4]{81}$.

Note that $3^4 = 81$ so $\sqrt[4]{81} = 3$.

Example 2

Find $\sqrt[3]{-125}$.

Since $(-5)^3 = -125$, we know that $\sqrt[3]{-125} = -5$.

Example 3

Simplify $\dfrac{5\sqrt[4]{243}}{\sqrt[4]{3}}$.

$$\dfrac{5\sqrt[4]{243}}{\sqrt[4]{3}}$$

$$= \dfrac{5\sqrt[4]{81}\sqrt[4]{3}}{\sqrt[4]{3}}$$

$$= \dfrac{5\sqrt[4]{81}}{1}$$

$$= 5 \cdot 3$$

$$= 15$$

Example 4

Expand $(\sqrt{3} + \sqrt{11})(\sqrt{3} - \sqrt{11})$.

$$(\sqrt{3} + \sqrt{11})(\sqrt{3} - \sqrt{11})$$
$$= 3 + \sqrt{3}\sqrt{11} - \sqrt{3}\sqrt{11} - 11$$
$$= 3 - 11$$
$$= -8$$

Example 5

Simplify $\dfrac{5}{\sqrt{2} - \sqrt{3}}$ by rationalizing the denominator.

$$\dfrac{5}{\sqrt{2} - \sqrt{3}}$$

$$= \dfrac{5}{\sqrt{2} - \sqrt{3}} \cdot \dfrac{\sqrt{2} + \sqrt{3}}{\sqrt{2} + \sqrt{3}}$$

$$= \dfrac{5(\sqrt{2} + \sqrt{3})}{(\sqrt{2} - \sqrt{3})(\sqrt{2} + \sqrt{3})}$$

$$= \dfrac{5(\sqrt{2} + \sqrt{3})}{2 - 3}$$

$$= -5(\sqrt{2} + \sqrt{3})$$

Example 6

Simplify $\sqrt[5]{-224}$.

Notice that $224 = 32 \cdot 7 = 2^5 7$, so we have $\sqrt[5]{-224} = \sqrt[5]{-32}\sqrt[5]{7} = -2\sqrt[5]{7}$.

Example 7

Simplify $10\sqrt[3]{2} + \sqrt[3]{54} - 7\sqrt[3]{16}$.

$$10\sqrt[3]{2} + \sqrt[3]{54} - 7\sqrt[3]{16}$$
$$= 10\sqrt[3]{2} + \sqrt[3]{27 \cdot 2} - 7\sqrt[3]{8 \cdot 2}$$
$$= 10\sqrt[3]{2} + 3\sqrt[3]{2} - 7 \cdot 2\sqrt[3]{2}$$
$$= 10\sqrt[3]{2} + 3\sqrt[3]{2} - 14\sqrt[3]{2}$$
$$= -\sqrt[3]{2}$$

Common Errors

- Forgetting that an nth root of a negative number can only be computed as a real number if n is odd.

Study Tips

- Knowing several facts about small powers of integers will make computing these roots very easy (and quick).

Problems

Simplify the following.

1. $\sqrt[3]{-500}$

2. $\sqrt[4]{405}\sqrt[4]{250}$

3. $\sqrt[5]{8}\sqrt[5]{12}$

4. $\dfrac{\sqrt[3]{648}}{\sqrt[3]{81}}$

5. $10\sqrt[5]{32} + 2\sqrt[5]{-243} - 5\sqrt[5]{-1}$

6. $\sqrt[4]{32} - 2\sqrt[4]{4802} - 3\sqrt[4]{512}$

Rationalize the denominators in the following.

7. $\dfrac{1}{\sqrt{5} + \sqrt{2}}$

8. $\dfrac{6}{\sqrt{31} - \sqrt{30}}$

9. $\dfrac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}}$

10. $\dfrac{5\sqrt{10}}{2\sqrt{10} - 4\sqrt{7}}$

Solving Equations Involving Radicals

Lesson 26

Topics in This Lesson

- Connecting radicals to rational exponents.

- Solving equations that involve radicals or rational exponents.

- Checking for extraneous solutions to equations.

Summary

In this lesson, we begin solving equations in which one or more of the terms involve a radical symbol or a rational power.

Definitions and Formulas

extraneous solution: A solution of a simplified version of an equation that is not a solution of the original equation.

Examples

Example 1

Solve $\sqrt[3]{x+3} = \sqrt[3]{5x-9}$.

We begin by cubing both sides of the equation.

$$\sqrt[3]{x+3} = \sqrt[3]{5x-9}$$
$$(\sqrt[3]{x+3})^3 = (\sqrt[3]{5x-9})^3$$
$$((x+3)^{1/3})^3 = ((5x-9)^{1/3})^3$$
$$x+3 = 5x-9$$

Now the equation is just a linear equation, which we can solve quickly.

$$x+3 = 5x-9$$
$$3 = 4x-9$$
$$12 = 4x$$
$$x = 3$$

Let's check our solution.

$$\sqrt[3]{x+3} = \sqrt[3]{5x-9}$$
$$\sqrt[3]{3+3} = \sqrt[3]{5(3)-9}$$
$$\sqrt[3]{6} = \sqrt[3]{15-9}$$
$$\sqrt[3]{6} = \sqrt[3]{6}$$

Since this last statement is true, we know that $x = 3$ is indeed a solution.

Example 2

Solve $3x = 2x + \sqrt{4x+5}$.

We begin by isolating the radical symbol.

$$3x = 2x + \sqrt{4x+5}$$
$$x = \sqrt{4x+5}$$

Next we square both sides of the equation.

$$x^2 = 4x+5$$

Now we solve this quadratic equation.

$$x^2 = 4x+5$$
$$x^2 - 4x - 5 = 0$$
$$(x-5)(x+1) = 0$$

Our solutions appear to be $x = 5$ and $x = -1$, but we must check to confirm that neither of these is an extraneous solution of the original equation.

$$3x = 2x + \sqrt{4x+5}$$
$$3(5) = 2(5) + \sqrt{4(5)+5}$$
$$15 = 10 + \sqrt{25}$$
$$15 = 10 + 5$$
$$15 = 15$$

This shows that $x = 5$ really is a solution. Now we check $x = -1$.

$$3x = 2x + \sqrt{4x+5}$$
$$3(-1) = 2(-1) + \sqrt{4(-1)+5}$$
$$-3 = -2 + \sqrt{1}$$
$$-3 = -2 + 1$$
$$-3 = -1$$

So $x = -1$ is not a solution of the original equation. This means that the only solution of the original equation is $x = 5$.

Example 3

Solve $\sqrt[5]{x^5 + 6x^2 - 54} = x$ for the variable x.

We begin by raising both sides of the equation to the fifth power.

$$\sqrt[5]{x^5 + 6x^2 - 54} = x$$
$$(x^5 + 6x^2 - 54)^{1/5} = x$$
$$((x^5 + 6x^2 - 54)^{1/5})^5 = x^5$$
$$x^5 + 6x^2 - 54 = x^5$$
$$6x^2 - 54 = 0$$

We now solve this equation.

$$6x^2 - 54 = 0$$
$$6(x^2 - 9) = 0$$
$$x^2 - 9 = 0$$
$$(x-3)(x+3) = 0$$

So our possible solutions are $x = 3$ and $x = -3$. Let's check them both.

$$\sqrt[5]{x^5 + 6x^2 - 54} = x$$
$$\sqrt[5]{3^5 + 6(3)^2 - 54} = 3$$
$$\sqrt[5]{243 + 54 - 54} = 3$$
$$\sqrt[5]{243} = 3$$
$$3 = 3$$

So $x = 3$ really is a solution. Now let's check $x = -3$.

$$\sqrt[5]{x^5 + 6x^2 - 54} = x$$
$$\sqrt[5]{(-3)^5 + 6(-3)^2 - 54} = -3$$
$$\sqrt[5]{-243 + 54 - 54} = -3$$
$$\sqrt[5]{-243} = -3$$
$$-3 = -3$$

So $x = -3$ is also a solution.

Common Errors

- Failing to check your solutions! If you forget to check your solutions, you may include some that are not correct. Extraneous solutions arise very easily in these problems.

Study Tips

- Know the laws of exponents, and get comfortable using them.

- Memorize several arithmetic facts about powers; for example, knowing quickly that $\sqrt[5]{243} = 3$ is very helpful.

Problems

Solve the following equations.

1. $\sqrt[4]{1 - 5x} = \sqrt[4]{5 - x}$

2. $\sqrt[5]{3x - 10} = \sqrt[5]{2 - x}$

3. $\sqrt{90 - x} = x$

4. $\sqrt{8x - 12} + 5x = 6x$

5. $-4x - 2 = -5x + 3 + \sqrt{3x - 11}$

6. $14 = 34 - 5\sqrt[3]{2x}$

7. $\sqrt[3]{x^3 + x^2 + 13x + 42} = x$

8. $\sqrt[4]{x^4 + x^2 + 13x + 42} = x$

9. $\sqrt[5]{32x^5 + x^2 - 2x - 80} = 2x$

10. $\sqrt[6]{x^6 + x^3 + 2x^2 - x - 2} = x$

Graphing Power, Radical, and Root Functions
Lesson 27

Topics in This Lesson

- Determining the domain of a function that involves rational powers or radicals.

- Plotting graphs of functions that involve rational powers or radicals.

Summary

We consider power functions (which contain rational powers) and ask 2 fundamental questions: (1) What is the domain of each such function? (2) What does the graph of such a function look like?

Examples

Example 1

Sketch the graph of $f(x) = \sqrt{x-2} + 6$.

We see that the graph of $f(x)$ is the same as the graph of $\sqrt{x}$ except that it is shifted 2 units to the right and 6 units up. So the graph looks like this.
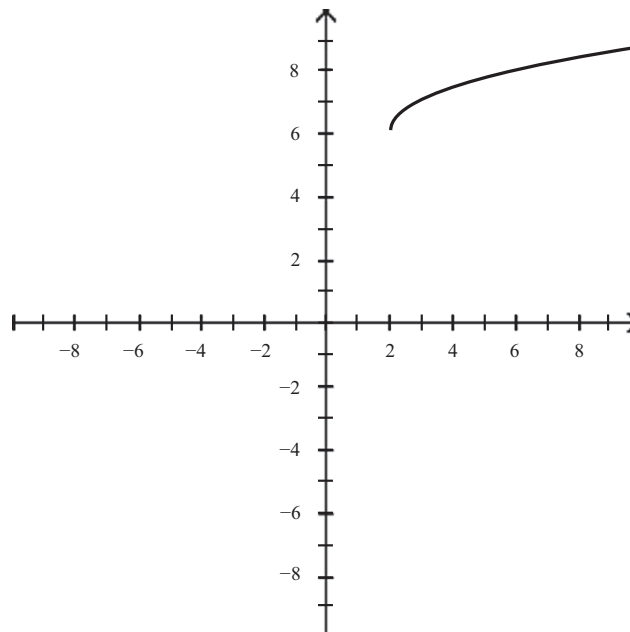
Example 2

Sketch the graph of $f(x) = x^{1/3}$.

First, note that the domain of this function is the set of all real numbers, not just the set of nonnegative real numbers. So the graph of this function is going to stretch all the way across the xy-plane, not just the right-hand side of the plane.

Next, we plot several points and sketch the graph.

$f(0) = 0$
$f(1) = 1$
$f(8) = 2$
$f(27) = 3$
$f(81) = 4$

$f(-1) = -1$
$f(-8) = -2$
$f(-27) = -3$
$f(-81) = -4$
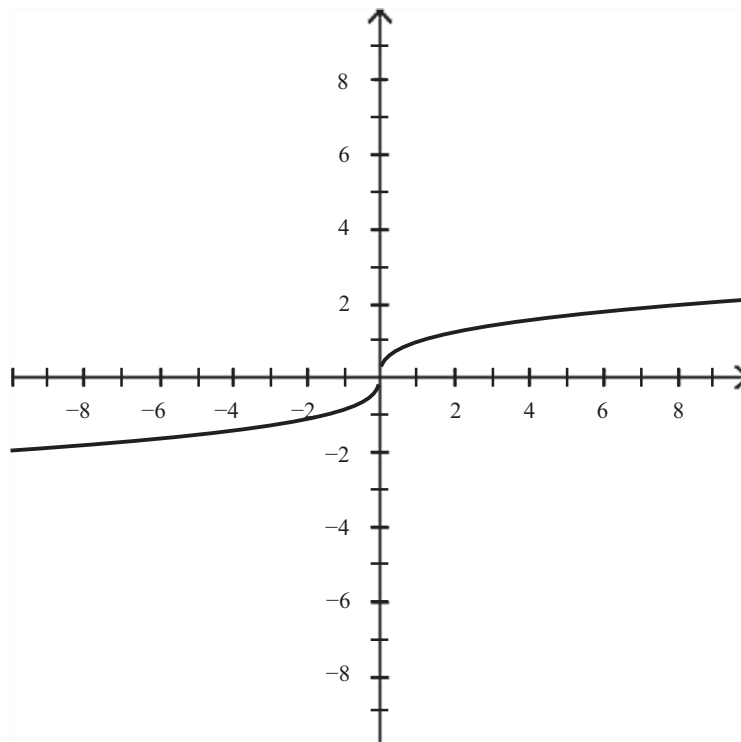
Example 3

Sketch the graph of $f(x) = x^{2/3} - 4$.

We know that the graph of $x^{2/3}$ is the following.
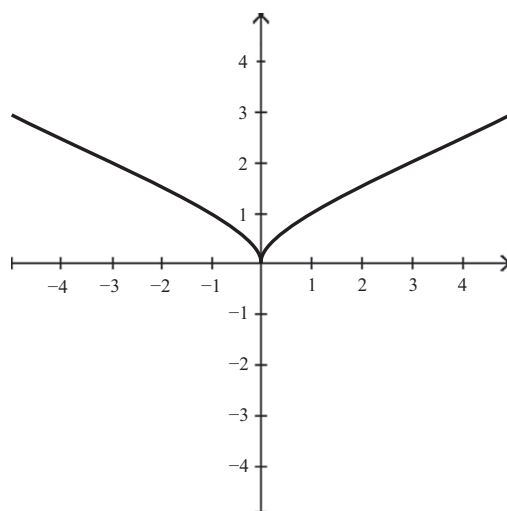
We shift this graph downward 4 units to find the graph for this example.
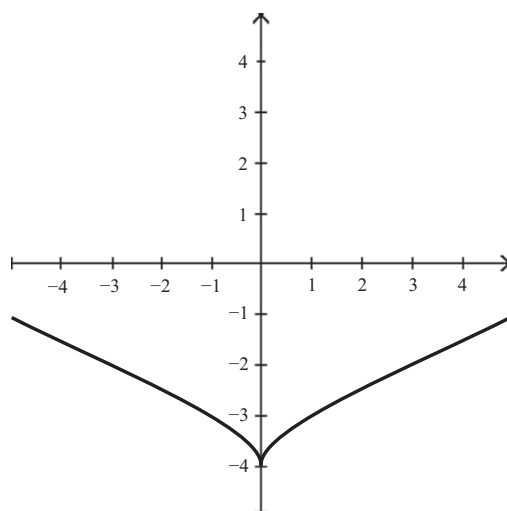
Common Errors

- Failing to correctly use the information about the domain of a function when sketching the graph of the function.

Study Tips

- Use graph paper!

- When in doubt, plot some points and connect the dots.

Problems

Find the domain of each function.

1. $f(x) = (x+3)^{1/4} + 2$

2. $f(x) = x^{1/5} - 1$

3. $f(x) = \sqrt[3]{x-1} + 6$

4. $f(x) = x^{4/7}$

5. $f(x) = -(x-2)^{1/3} + 4$

Sketch the graph of each of the following functions.

6. $f(x) = (x+3)^{1/4} + 2$

7. $f(x) = x^{1/5} - 1$

8. $f(x) = \sqrt[3]{x-1} + 6$

9. $f(x) = x^{4/7}$

10. $f(x) = -(x-2)^{1/3} + 4$

An Introduction to Rational Functions
Lesson 28

Topics in This Lesson

- The definition of a rational function.

- The domain of a rational function.

- Horizontal and vertical asymptotes.

- Graphs of rational functions.

Summary

We meet a new family of functions, called rational functions, that are basically ratios of 2 polynomial functions. We discuss the domain of a rational function and then draw the graphs of a variety of rational functions.

Definitions and Formulas

domain of a rational function: Let $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials. Then the domain of $f(x)$ is the set of all real numbers except for the values of x where $q(x) = 0$.

rational function: A ratio of 2 polynomial functions, or one polynomial divided by another polynomial.

Examples

Example 1

Determine the domain of the function $f(x) = \dfrac{x^2 - 4}{x^2 - 1}$.

We begin by factoring the numerator and denominator of the function.

$$f(x) = \frac{(x-2)(x+2)}{(x-1)(x+1)}$$

From here we see that the domain of this function is the set of all real numbers except the values $x = 1$ and $x = -1$, because these are the 2 values of x that cause the denominator to equal zero.

Example 2

Sketch the graph of $y = \dfrac{2x^2 - 8x + 6}{x^2 - x - 6}$.

We begin by factoring.

$$y = \frac{2x^2 - 8x + 6}{x^2 - x - 6}$$

$$y = \frac{2(x^2 - 4x + 3)}{x^2 - x - 6}$$

$$y = \frac{2(x-3)(x-1)}{(x-3)(x+2)}$$

$$y = \frac{2(x-1)}{(x+2)}$$

Something very important has happened. The $x - 3$ factors have cancelled out. But remember—the 2 numbers *not* in the domain of the original function are $x = -2$ and $x = 3$.

There is another very important fact we need to remember. Since $x - 3$ cancelled out, there is not a vertical asymptote at $x = 3$. Instead, there is just an open circle at $x = 3$. So there is just one vertical asymptote in this graph, at the line $x = -2$.

Now let's get all the information we can.

x-intercept:

$$2(x-1) = 0$$
$$x - 1 = 0$$
$$x = 1$$

The *x*-intercept is (1, 0).

y-intercept:

$$y = 2(0-1)/(0+2)$$
$$= -2/2$$
$$= -1$$

The *y*-intercept is (0, −1).

vertical asymptote:

$$x + 2 = 0$$
$$x = -2$$

The vertical asymptote is at $x = -2$. Remember: There is not a vertical asymptote at $x = 3$, just an open circle.

horizontal asymptote:

$$y = 2x^2 / x^2$$
$$= 2$$

The horizontal asymptote is at $y = 2$.

Write out these intercepts and asymptotes, plot several points, and connect the dots.

x	−5	−4	−3	−1	0	1	2
y	4	5	8	−4	−1	0	1/2
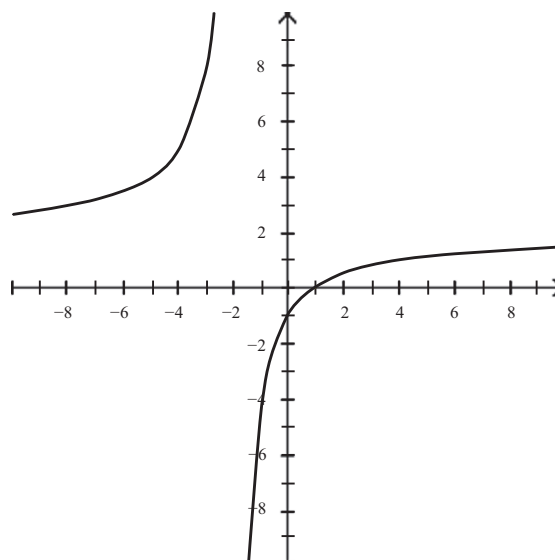
Example 3

Sketch the graph of $f(x) = \dfrac{3(x-1)(x+1)}{x(x-2)(x+2)}$.

The function is already in factored form, which helps us tremendously. Now let's see what we know about the graph.

First, the domain of the function is the set of all real numbers except $x = 0$, $x = 2$, and $x = -2$. Moreover, since none of the factors in the denominator cancel with any of the factors in the numerator, we know that there are vertical asymptotes at all 3 of these values.

Next, we should determine the intercepts and asymptotes.

x-intercepts:

The x-intercepts are $(1, 0)$ and $(-1, 0)$, which we know thanks to the factored form of the numerator.

y-intercept:

There is no y-intercept because the function is undefined at $x = 0$.

vertical asymptotes:

As noted above, the vertical asymptotes occur at the lines $x = -2$, $x = 0$, and $x = 2$.

horizontal asymptote:

The horizontal asymptote occurs at $y = 0$ because the degree of the numerator (which is 2) is less than the degree of the denominator (which is 3).

Plotting some additional points, and keeping in mind all the information summarized above, we see that the graph looks like the following.
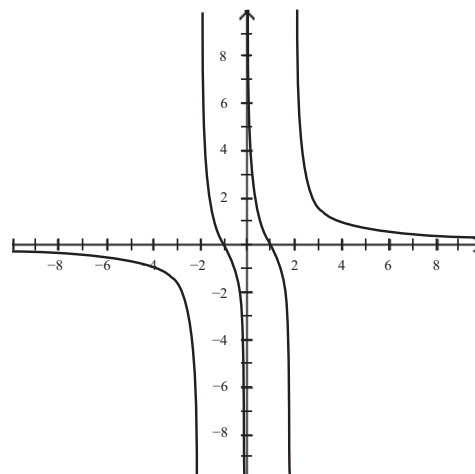
Common Errors

- When finding the domain of a rational function, you may accidentally exclude those values of x that cause the *numerator* to equal zero. Remember: We only exclude those values of x that cause the *denominator* to equal zero.

Study Tips

- Be sure you carefully gather all the information you can about the graph before trying to sketch it—intercepts, asymptotes, and several sample points. These problems can take some time, especially when you are trying to interpret all the information, so be patient.

- Watch for cancellation of factors in the numerator and denominator of the original function. If a factor does cancel, then it might mean that the graph has a hole rather than a vertical asymptote at a certain value.

Problems

Find the domain, horizontal asymptote, and vertical asymptote(s) of each of the following rational functions.

1. $f(x) = \dfrac{2x+1}{x-1}$

2. $f(x) = 2 + \dfrac{7}{x+3}$

3. $f(x) = \dfrac{-6}{x^2 - 4x}$

4. $f(x) = \dfrac{4x^2 - 16x}{x^2 - 2x - 3}$

5. $f(x) = \dfrac{5x^2 + 25x + 30}{x^2 + 3x + 2}$

Sketch the graph of each of these rational functions.

6. $f(x) = \dfrac{2x+1}{x-1}$

7. $f(x) = 2 + \dfrac{7}{x+3}$

8. $f(x) = \dfrac{-6}{x^2 - 4x}$

9. $f(x) = \dfrac{4x^2 - 16x}{x^2 - 2x - 3}$

10. $f(x) = \dfrac{5x^2 + 25x + 30}{x^2 + 3x + 2}$

The Algebra of Rational Functions
Lesson 29

Topics in This Lesson

- Combining rational functions using addition, subtraction, multiplication, division, and composition.

Summary

The goal of this lesson is straightforward—to work through a variety of problems involving the addition, subtraction, multiplication, division, and composition of rational functions, which will allow us to build new rational functions out of old ones.

Examples

Example 1

Simplify $\dfrac{x^3 - x^2 - 6x}{x^2 + 5x + 6}$.

Notice that there is an x in all 3 terms of the numerator, so an x can be factored out. That leaves us with

$$\frac{x(x^2 - x - 6)}{x^2 + 5x + 6} \, .$$

After additional factoring, we find the original rational expression can be rewritten as below.

$$\frac{x(x-3)(x+2)}{(x+3)(x+2)}$$
$$= \frac{x(x-3)}{x+3}$$

Example 2

Simplify $\dfrac{x}{x^2 - 9} - \dfrac{x}{x^2 + 6x + 9}$.

We factor the 2 denominators.

$$x^2 - 9 = (x - 3)(x + 3)$$
$$x^2 + 6x + 9 = (x + 3)^2$$

That means that the least common denominator is $(x - 3)(x + 3)^2$. So our problem can be rewritten as follows.

$$\frac{x}{x^2 - 9} - \frac{x}{x^2 + 6x + 9}$$

$$= \frac{x}{(x - 3)(x + 3)} - \frac{x}{(x + 3)^2}$$

$$= \frac{x(x + 3)}{(x - 3)(x + 3)^2} - \frac{x(x - 3)}{(x - 3)(x + 3)^2}$$

$$= \frac{x^2 + 3x - x^2 + 3x}{(x - 3)(x + 3)^2}$$

$$= \frac{6x}{(x - 3)(x + 3)^2}$$

Example 3

Multiply $\dfrac{3x + 7}{5x + 10}$ by $x^2 + 7x + 10$.

$$\frac{3x + 7}{5x + 10} \times (x^2 + 7x + 10)$$

$$= \frac{3x + 7}{5x + 10} \times \frac{x^2 + 7x + 10}{1}$$

$$= \frac{3x + 7}{5(x + 2)} \times \frac{(x + 2)(x + 5)}{1}$$

$$= \frac{(3x + 7)(x + 5)}{5}$$

Example 4

Divide $\dfrac{x^2+13x+40}{x-4}$ by $\dfrac{x+8}{x^2-16}$.

$$\frac{x^2+13x+40}{x-4}\div\frac{x+8}{x^2-16}$$

$$=\frac{x^2+13x+40}{x-4}\times\frac{x^2-16}{x+8}$$

$$=\frac{(x+8)(x+5)(x-4)(x+4)}{(x-4)(x+8)}$$

$$=\frac{(x+5)(x+4)}{1}$$

$$=(x+5)(x+4)$$

Example 5

Let $f(x)=\dfrac{7x+1}{x+4}$ and $g(x)=\dfrac{-4x+1}{x-7}$. Find $f(g(x))$.

$$f(g(x))=f\left(\frac{-4x+1}{x-7}\right)$$

$$=\frac{7\left(\dfrac{-4x+1}{x-7}\right)+1}{\dfrac{-4x+1}{x-7}+4}$$

$$=\frac{\left(\dfrac{-28x+7+x-7}{x-7}\right)}{\dfrac{-4x+1+4x-28}{x-7}}$$

$$=\frac{-28x+7+x-7}{-4x+1+4x-28}$$

$$=\frac{-27x}{-27}$$

$$=x$$

Common Errors

- Forgetting to distribute minus signs when subtracting.

- Taking the reciprocal of the first rational expression rather than the second rational expression when converting from a division problem to a multiplication problem.

Study Tips

- Be deliberate about your algebraic simplifications and arithmetic. These problems sometimes involve many steps, and errors can crop up easily.

Problems

Simplify the following.

1. $\dfrac{2}{x^2 - 9x + 14} + \dfrac{5}{x^2 - 12x + 20}$

2. $7 + \dfrac{x-1}{x-3}$

3. $\dfrac{x+5}{x-5} - \dfrac{x+2}{x-2}$

4. $\dfrac{3x}{x^2 - 16} - \dfrac{2x+1}{x^2 + 6x + 8}$

5. $\dfrac{x^2 - 16}{x+9} \cdot \dfrac{x^2 - x - 90}{x^2 + 19x + 60}$

6. $\dfrac{x^3 - 13x^2 + 42x}{x^2 - 49} \cdot \left(x^2 + 8x + 7 \right)$

7. $\dfrac{x+3}{x+2} \div \dfrac{x^2 + 2x - 3}{x^2 - 2x + 1}$

8. $\dfrac{x^2 + x - 56}{x^2 - 2x - 80} \div \dfrac{1}{x+8}$

Determine the composition function $f(g(x))$ for the following pairs of functions.

9. $f(x) = \dfrac{x}{x+1}$, $g(x) = \dfrac{x}{x-1}$

10. $f(x) = \dfrac{2}{x}$, $g(x) = \dfrac{2x}{x^2 - 1}$

Partial Fractions
Lesson 30

Topics in This Lesson

- Partial fractions.

Summary

We often combine fractions or rational expressions into one expression. But there are also times when we wish to do the opposite—to split a more complicated rational expression into the sum of simpler rational expressions. This is accomplished by the method of partial fractions.

Definitions and Formulas

The **method of partial fractions** involves splitting apart rational expressions so that the denominators are "separated" from one another. For example, it is true that

$$\frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x},$$

which you can prove by finding the lowest common denominator for the 2 fractions on the right-hand side of the equation and then performing the subtraction.

basic method of partial fractions: If you have a fraction with only linear terms in the denominator (terms raised to just the first power, like x or $x + 2$), or if you can factor a polynomial in the denominator into unrepeated linear terms, then you can split the fraction into the sum of 2 fractions of the form $\frac{A}{x} + \frac{B}{x+2}$, where A and B are constants. Next, find the lowest common denominator of your new fractions, recombine them, and then solve for the constants A and B.

repeated linear term in the denominator: If you have a repeated linear term in the denominator, such as $(x-1)^2$, then you need to add 2 terms to deal with it: 1 for its linear term and 1 for the linear term squared. That is, you will need to include $\frac{A}{x-1} + \frac{B}{(x-1)^2}$ in the partial fraction decomposition, where A and B are just constants. Then proceed as before: Recombine the fractions and solve for the constants.

irreducible quadratic term in the denominator: What if you have a polynomial in the denominator that is irreducible (or will not factor), like $x^2 + 1$? Then in the partial fraction decomposition, you must insert a linear term to the numerator of your new fraction, as below.

$$\frac{Ax + B}{x^2 + 1}$$

As before, A and B are just constants. Then the process continues as before.

Examples

Example 1

Find the partial fraction decomposition of $\dfrac{2x + 1}{x^2 + 2x - 3}$.

The first thing we need to do is factor the denominator.

$$\frac{2x + 1}{(x - 1)(x + 3)}$$

We then split this expression into 2 rational expressions, each of which simply contains a constant in the numerator.

$$\frac{2x + 1}{(x - 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 3}$$

Now we combine the sum on the right with a common denominator and then solve for A and B.

$$\frac{A}{x - 1} + \frac{B}{x + 3}$$

$$= \frac{A(x + 3)}{(x - 1)(x + 3)} + \frac{B(x - 1)}{(x - 1)(x + 3)}$$

$$= \frac{Ax + 3A + Bx - B}{(x - 1)(x + 3)}$$

$$= \frac{Ax + Bx + 3A - B}{(x - 1)(x + 3)}$$

We know that this last quantity equals $\dfrac{2x + 1}{(x - 1)(x + 3)}$. Since the denominators are equal, the numerators must be equal.

$$Ax + Bx + 3A - B = 2x + 1$$

So we know the following.

$$A + B = 2$$
$$3A - B = 1$$

This means we have 2 equations with 2 unknowns, so we can solve the system. From the first equation, we know that $B = 2 - A$. Now substitute this quantity for B in the other equation.

$$3A - B = 1$$
$$3A - (2 - A) = 1$$
$$3A - 2 + A = 1$$
$$4A - 2 = 1$$
$$4A = 3$$
$$A = \frac{3}{4}$$

Then we know the following.

$$B = 2 - A$$
$$B = 2 - \frac{3}{4}$$
$$B = \frac{5}{4}$$

So we have our decomposition.

$$\frac{2x + 1}{(x - 1)(x + 3)} = \frac{\frac{3}{4}}{x - 1} + \frac{\frac{5}{4}}{x + 3}$$

Example 2

Find the partial fraction decomposition of $\dfrac{2}{x^3 + x}$.

We start by factoring the denominator and writing the decomposition with unknowns in the numerator.

$$\frac{2}{x^3 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

Now we proceed as we did in the previous example.

$$\frac{A}{x}+\frac{Bx+C}{x^2+1}$$

$$=\frac{A(x^2+1)}{x(x^2+1)}+\frac{(Bx+C)x}{x(x^2+1)}$$

$$=\frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}$$

So we know the following.

$$\frac{2}{x^3+x}=\frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}$$

Since the denominators are equal, we can set numerators equal to get the following.

$$A+B=0$$
$$C=0$$
$$A=2$$

Remember, the numerator of the original expression is 2, which we can think of as $0x^2+0x+2$.

So we know that $A=2, B=-2,$ and $C=0$. Therefore, we know our full partial fraction decomposition.

$$\frac{2}{x^3+x}=\frac{2}{x}+\frac{-2x+0}{x^2+1}$$

$$\frac{2}{x^3+x}=\frac{2}{x}-\frac{2x}{x^2+1}$$

Example 3

Find the partial fraction decomposition of $\dfrac{x+3}{(x-1)^2}$.

We see a repeated linear factor in the denominator.

$$\frac{x+3}{(x-1)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}$$

We proceed as we have done in previous examples.

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}$$

$$=\frac{A(x-1)}{(x-1)^2}+\frac{B}{(x-1)^2}$$

$$=\frac{Ax-A}{(x-1)^2}+\frac{B}{(x-1)^2}$$

$$=\frac{Ax-A+B}{(x-1)^2}$$

So we know the following.

$$A=1$$
$$-A+B=3$$

Therefore, $B=4$. So our partial fraction decomposition is

$$\frac{x+3}{(x-1)^2}=\frac{1}{x-1}+\frac{4}{(x-1)^2}\ .$$

Common Errors

- Making small arithmetic mistakes when solving for the coefficients in systems of linear equations.

Study Tips

- Think carefully about the initial setup of the partial fraction decomposition. It is important that the correct numerators be chosen for the decomposition. If incorrectly chosen, they may lead to either incorrect solutions or no solutions at all, and this could prove frustrating given the time taken to complete these problems.

Lesson 30: Partial Fractions

Problems

State the form of the partial fraction decomposition of each of these rational expressions.

1. $\dfrac{5x+7}{x^2+2x-3}$

2. $\dfrac{3x+8}{(x+2)^2}$

3. $\dfrac{2x}{(x+1)(x^2+1)}$

4. $\dfrac{36}{x^2(x-3)}$

5. $\dfrac{7x^2-16x+36}{x^4-16}$

Find the partial fraction decomposition of each of these rational expressions.

6. $\dfrac{5x+7}{x^2+2x-3}$

7. $\dfrac{3x+8}{(x+2)^2}$

8. $\dfrac{2x}{(x+1)(x^2+1)}$

9. $\dfrac{36}{x^2(x-3)}$

10. $\dfrac{7x^2-16x+36}{x^4-16}$

An Introduction to Exponential Functions
Lesson 31

Topics in This Lesson

- The definition of an exponential function.

- Graphs of exponential functions.

- The difference between exponential functions that grow and those that decay.

Summary

In this lesson, we transition to a new family of functions—exponential functions. These are very different from the functions we have previously studied. Exponential functions are naturally beneficial in a number of real-world calculations, including those of population growth, compound interest in a bank account, the net worth of a car or piece of equipment (due to depreciation), the pH of a chemical solution, or the intensity of an earthquake.

Definitions and Formulas

base (of a logarithm): The value b of the exponential function.

decay factor: The specific term for the b value of the exponential function when $b < 1$.

exponential function: A function of the form $f(x) = ab^x$, where x is a real number, a is a nonzero constant, and b is a positive real number that is not equal to 1.

growth factor: The specific term for the b value on the exponential function when $b > 1$.

Examples

Example 1

Sketch the graph of $f(x) = 2^x$ by computing several points and connecting the dots.

I suggest that we find $f(x)$ for $x = -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$ to see what's happening.

$$f(0) = 2^0 = 1$$

So we know that the graph goes through the point (0, 1); that's the *y*-intercept. Let's keep moving.

$$f(1) = 2^1 = 2$$
$$f(2) = 2^2 = 4$$
$$f(3) = 2^3 = 8$$
$$f(4) = 2^4 = 16$$
$$f(5) = 2^5 = 32$$

That means that the following points are on the graph of *f*(*x*): (0, 1), (1, 2), (2, 4), (3, 8), (4, 16), and (5, 32).

Next, we see the following.

$$f(-1) = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$$
$$f(-2) = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$$
$$f(-3) = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$
$$f(-4) = 2^{-4} = \frac{1}{2^4} = \frac{1}{16}$$
$$f(-5) = 2^{-5} = \frac{1}{2^5} = \frac{1}{32}$$

We plot these points on a sheet of graph paper and connect the dots, getting the following.
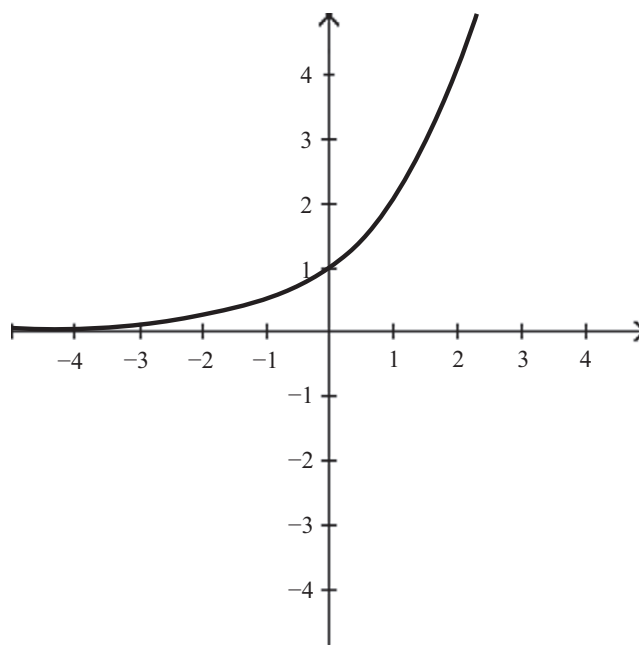
Example 2

Find an exponential function $y = ab^x$ for a graph that includes the 2 points (3, 24) and (2, 12).

If the graph passes through (3, 24), it means that $x = 3$ and $y = 24$ must make the equation true. So that means that

$$24 = ab^3.$$

Next, if the graph passes through (2, 12), it means that $x = 2$ and $y = 12$ also must make the equation true. So

$$12 = ab^2.$$

Now this second equation can be solved for a.

$$a = \frac{12}{b^2}$$

Plug that in for a in the first equation.

$$24 = ab^3$$
$$24 = \left(\frac{12}{b^2}\right)b^3$$
$$24 = 12b$$
$$b = 2$$

Now just replace b by 2 in either of the first 2 equations.

$$24 = ab^3$$
$$24 = a(2)^3$$
$$24 = 8a$$
$$a = 3$$

So the function in question is $y = 3 \cdot 2^x$.

Example 3

Sketch the graph of $f(x) = \left(\frac{1}{3}\right)^x$.

Let's plot several points and connect the dots.

$$f(-5) = \left(\frac{1}{3}\right)^{-5} = 3^5 = 243$$

$$f(-4) = \left(\frac{1}{3}\right)^{-4} = 3^4 = 81$$

$$f(-3) = \left(\frac{1}{3}\right)^{-3} = 3^3 = 27$$

$$f(-2) = \left(\frac{1}{3}\right)^{-2} = 3^2 = 9$$

$$f(-1) = \left(\frac{1}{3}\right)^{-1} = 3^1 = 3$$

$$f(0) = \left(\frac{1}{3}\right)^{0} = 1$$

$$f(1) = \left(\frac{1}{3}\right)^{1} = \frac{1}{3}$$

$$f(2) = \left(\frac{1}{3}\right)^{2} = \frac{1}{9}$$

$$f(3) = \left(\frac{1}{3}\right)^{3} = \frac{1}{27}$$

$$f(4) = \left(\frac{1}{3}\right)^{4} = \frac{1}{81}$$

$$f(5) = \left(\frac{1}{3}\right)^{5} = \frac{1}{243}$$

Plotting these points gives the following.
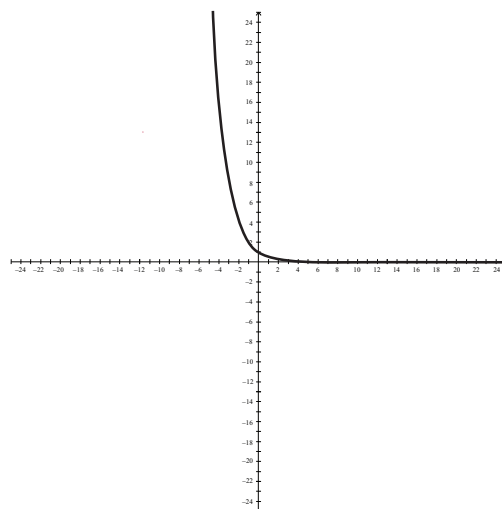
Note that this is the mirror image of the graph of $f(x) = 3^x$.

Common Errors

- Confusing the laws of exponents.

Study Tips

- Memorize several exponential facts (like $4^3 = 64$). Knowing many of these facts off the top of your head can help with these kinds of problems.

Problems

Sketch the graph of each of the following exponential functions.

1. $y = 5^x$

2. $y = \left(\dfrac{1}{5}\right)^x$

3. $y = 2^{x+6}$

4. $y = 4^{x-2} + 3$

5. $y = \left(\dfrac{1}{4}\right)^x - 3$

6. $y = -7^x + 2$

Find an exponential function of the form $y = ab^x$ that goes through each pair of given points.

7. $\left(1, \dfrac{5}{8}\right), \left(3, \dfrac{125}{8}\right)$

8. $(0, 7), (-2, 28)$

9. $(1, -6), (4, -48)$

10. $\left(2, \dfrac{1}{36}\right), \left(-1, \dfrac{3}{4}\right)$

An Introduction to Logarithmic Functions
Lesson 32

Topics in This Lesson

- An introduction to logarithmic functions.

- Properties of logarithms.

- Graphs of logarithms.

Summary

In the previous lesson, we discussed exponential functions. We now look at a closely related family of functions, logarithmic functions. Logarithmic functions are actually inverses of exponential functions. We consider properties satisfied by logarithms as well as graphs of logarithmic functions.

Definitions and Formulas

logarithm: If $y = b^x$, then the logarithm to the base b of a positive number y is denoted by $\log_b y$ and is defined by $\log_b y = x$.

properties of logarithms: Let M, N, and b be positive numbers with $b \neq 1$, and let x be any real number.

    1. $\log_b(MN) = \log_b M + \log_b N$

    2. $\log_b\left(\dfrac{M}{N}\right) = \log_b M - \log_b N$

    3. $\log_b(M^x) = x\log_b M$

Examples

Example 1

Evaluate $\log_3 81$.

Since $3^4 = 81$, we know that $\log_3 81 = 4$.

Example 2

Sketch the graph of $f(x) = \log_3 x$.

We begin by evaluating the function at numerous values of x. We choose those values of x wisely in order to make these evaluations straightforward.

$$f(1) = \log_3(1) = 0 \text{ (because } 3^0 = 1)$$

$$f(3) = \log_3(3) = 1$$

$$f(9) = \log_3(9) = 2$$

$$f(27) = \log_3(27) = 3$$

$$f(81) = \log_3(81) = 4$$

Plotting these points gives us this sketch.
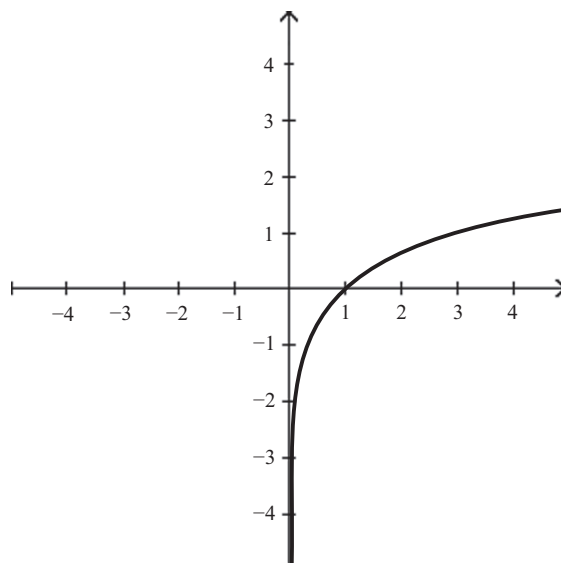
Notice that it has an x-intercept at (1, 0) and a vertical asymptote at the line $x = 0$.

Example 3

Write $3\log_4 t + \log_4 w - 5\log_4 z$ as a single logarithm.

The properties of logarithms can be extremely helpful to us when we're simplifying expressions involving logarithms.

$$3\log_4 t + \log_4 w - 5\log_4 z$$
$$= \log_4 t^3 + \log_4 w - \log_4 z^5$$
$$= \log_4 t^3 w - \log_4 z^5$$
$$= \log_4 \left(\frac{t^3 w}{z^5} \right)$$

Common Errors

- Confusing the statements of the properties. In particular, some students incorrectly think that $\log_b(M + N) = \log_b M \log_b N$.

Study Tips

- Memorize the properties of logarithms carefully and be sure you can apply them correctly.

Problems

Evaluate the following logarithms.

1. $\log_5 625$

2. $\log_8 4$

3. $\log_3 \left(\dfrac{1}{81} \right)$

Expand each logarithm using the properties of logarithms.

4. $\log_7 \left(\dfrac{x^2 y^3}{t^5 z} \right)$

5. $\log_2 \left(\dfrac{8a^5}{10b^3} \right)$

Combine each of the following into one logarithm.

6. $-3\log_4(x) + 5\log_4(y) - \log_4(z)$

7. $\log_5 x \cdot \log_5 y$

Sketch the graph of each of the following.

8. $y = \log_2(x-1)$

9. $y = \log_5(x+3) + 2$

10. $y = -\log_3(x)$

Uses of Exponential and Logarithmic Functions
Lesson 33

Topics in This Lesson

- The change of base formula.

- Solving equations that contain exponential and logarithmic functions.

- Solving problems that involve continuous compounded interest (the pert formula).

Summary

Exponential and logarithmic functions arise in a wide variety of equations related to real-world problems. We deal with some of these kinds of problems in this lesson.

Definitions and Formulas

change of base formula: For any positive numbers M, b, and c, where $b \neq 1$ and $c \neq 1$,

$$\log_b M = \frac{\log_c M}{\log_c b}.$$

pert formula: $A = Pe^{rt}$, where A is the amount of money in your account at any time t, P is the amount of principal (the amount you put in initially, or when $t = 0$), r is the interest rate, and t is time (in years). This formula only applies when the interest on the account is compounded continuously.

Examples

Example 1

Solve $3^{2x+1} = 144$.

$$
\begin{aligned}
3^{2x+1} &= 144 \\
\log_3(3^{2x+1}) &= \log_3(144) \\
(2x+1)\log_3(3) &= \log_3(144) \\
(2x+1)\cdot 1 &= \log_3(144) \\
2x+1 &= \log_3(144) \\
2x &= \log_3(144) - 1 \\
x &= \frac{\log_3(144) - 1}{2}
\end{aligned}
$$

If we want to find an approximation of this value, we need to convert it to either a common logarithm (log base 10) or a natural logarithm (log base e), as most calculators only have buttons for these 2 types of logarithms. Let's choose to convert this to common logarithms (which are often written without a subscript). Thanks to the change of base formula, we have

$$x = \frac{\log_3(144) - 1}{2}$$

$$= \frac{\dfrac{\log 144}{\log 3} - 1}{2}.$$

Now we can carefully use a calculator to estimate this solution: $x \approx 1.76186$.

Example 2

Solve $2\log x + \log 4 = 2$.

$$2\log x + \log 4 = 2$$
$$\log(x^2) + \log 4 = 2$$
$$\log(4x^2) = 2$$
$$10^{\log(4x^2)} = 10^2$$
$$4x^2 = 100$$
$$x^2 = 25$$
$$x = \pm 5$$

Now we must check our answers, just in case we have brought in any extraneous solutions.

$$2\log 5 + \log 4 = 2$$
$$\log 5^2 + \log 4 = 2$$
$$\log(5^2 \cdot 4) = 2$$
$$\log 100 = 2$$
$$2 = 2$$

So $x = 5$ is a solution.

$$2\log(-5) + \log 4 = 2$$

But this equation makes no sense! Remember, the domain of a log function is the set of positive real numbers, so $\log(-5)$ is not defined. Therefore, $x = -5$ is not a solution. That means our only solution is $x = 5$.

Example 3

An initial investment of $1000 is now valued at $1750.67. The interest rate is 8%, compounded continuously. How long has the money been invested in this account?

Since the interest is being compounded continuously, we use the pert formula: $A = Pe^{rt}$. The initial investment amount of $1000 is the principal, so $P = 1000$. At this time, the amount in the account is $1750.67, and that is what is represented by A, so $A = 1750.67$. Lastly, the interest rate is 8%. We convert this to the decimal number 0.08, so $r = 0.08$. We now plug all this information into the formula.

$$A = Pe^{rt}$$
$$1750.67 = 1000e^{0.08t}$$
$$1.75067 = e^{0.08t}$$
$$\ln 1.75067 = 0.08t$$
$$\frac{\ln 1.75067}{0.08} = t$$

Using a calculator, we find that $t \approx 6.99998$. So the money has been in the account for about 7 years.

Example 4

An accountant realizes that one of his client's bank accounts now has $3795 in it. He knows that the bank has been continuously compounding interest on the account for 30 years at a rate of 7.5%. But he doesn't know how much money the client had when he started the account. He also knows that the client hasn't touched this account since starting it 30 years ago. So how much money did the client use to start that account?

Since the interest is being compounded continuously, we know that the pert formula is the one to use. A is the amount of money in the account now, so $A = 3795$. P is the principal, the initial amount in the account, which is what we're trying to solve for. We know that $r = 0.075$. And t is the number of years the account has been active, so $t = 30$. Plugging all this into the pert formula gives us the below.

$$3795 = Pe^{(0.075)(30)}$$
$$3795 = Pe^{2.25}$$
$$3795 = P(9.4877)$$
$$P = \frac{3795}{9.4877}$$
$$P \approx 399.99$$

That means that the client started the account with about $400.

Common Errors

- Errors when using a calculator to estimate these values. Care should be taken in making these calculations.

- Failing to memorize the change of base formula correctly.

Study Tips

- Continue to practice using the properties of logarithms.

Problems

Solve the following equations.

1. $11^{x-8} - 5 = 42$

2. $7 \cdot 6^{3x} = 42$

3. $10e^{2x-10} - 4 = 70$

4. $5^{4x+1} = 100$

5. $\log_3(x^2 + 8) - \log_3 4 = 3$

6. $\ln(x+7) + \ln(x+3) = \ln 77$

7. $-3\log_5(x+1) = -12$

8. $\log(x^2 + 9) = \log(7x - 3)$

9. An initial investment of $2000 was made 5 years ago. The interest rate is 6.5%, compounded continuously. What is the balance in the account now?

10. An initial investment of $5000 is now valued at $9110.59. The interest rate is 5%, compounded continuously. How long has the money been invested in this account?

The Binomial Theorem
Lesson 34

Topics in This Lesson

- Pascal's triangle.

- The binomial theorem.

- The formula for each value in Pascal's triangle, often denoted $C(n,r)$.

- The factorial function.

- Expanding the powers of a binomial.

Summary

In this lesson, we provide a connection between Pascal's triangle and powers of binomials after they have been expanded. We introduce the factorial function and discuss a formula for computing each value of Pascal's triangle independent of the others.

Definitions and Formulas

binomial theorem (special case): Let n be a positive integer. Then $(x+1)^n = C(n,0)x^n + C(n,1)x^{n-1} + C(n,2)x^{n-2} + C(n,n-1)x + C(n,n)$.

factorial function: For a positive integer n, the function "n factorial" (which is denoted $n!$) is the product of all the integers from 1 to n.

formula for elements of Pascal's triangle: Let $C(n, r)$ be the element in Pascal's triangle in the nth row and r^{th} column. For any $n \geq 0$ and $r \geq 0$, $C(n,r) = \dfrac{n!}{r!(n-r)!}$.

```
                          1
                       1     1
                    1     2     1
                 1     3     3     1
              1     4     6     4     1
           1     5    10    10     5     1
        1     6    15    20    15     6     1
     1     7    21    35    35    21     7     1
   1     8    28    56    70    56    28     8     1
 1     9    36    84   126   126    84    36     9     1
1    10    45   120   210   252   210   120    45    10     1
```

Pascal's Triangle

Examples

Example 1

Simplify $\dfrac{5!}{3!2!}$.

$$\frac{5!}{3!2!} = \frac{120}{6 \cdot 2} = \frac{120}{12} = 10$$

Example 2

Calculate $C(8, 6)$.

$$C(8,6) = \frac{8!}{6!2!} = \frac{8 \cdot 7 \cdot 6!}{6! \cdot 2!} = \frac{8 \cdot 7}{2} = \frac{56}{2} = 28$$

Example 3

Expand $(x+1)^{10}$.

Thanks to the binomial theorem, we can use the 10th row of Pascal's triangle to find these coefficients.

$$(x+1)^{10} = x^{10} + 10x^9 + 45x^8 + 120x^7 + 210x^6 + 252x^5 + 210x^4 + 120x^3 + 45x^2 + 10x + 1$$

Example 4

Calculate $C(100, 4)$.

$$C(100,4) = \frac{100!}{4!96!} = \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96!}{4!96!} = \frac{100 \cdot 99 \cdot 98 \cdot 97}{4!} = \frac{100 \cdot 99 \cdot 98 \cdot 97}{24} = 3921225$$

Common Errors

- Forgetting that the numbering of the rows and columns of Pascal's triangle starts at 0, not at 1.

Study Tips

- Practice computing values of $C(n,r)$, especially with the cancellations of the various factorial functions. These values will play an important role in future lessons.

Problems

Simplify the following.

1. $\dfrac{12!}{10!}$

2. $\dfrac{16!}{13!}$

Calculate the following.

3. $C(12, 9)$

4. $C(12, 3)$

5. $C(50, 50)$

6. $C(25, 24)$

7. $C(9, 6)$

Expand the following binomials using the binomial theorem.

8. $(x+1)^6$

9. $(x+1)^8$

10. $(2t+1)^7$

Permutations and Combinations
Lesson 35

Topics in This Lesson

- Permutations.

- Combinations.

- The multiplication principle.

Summary

In this lesson, we begin our discussion of how to use permutations and combinations to solve problems that involve counting objects.

Definitions and Formulas

combination: A selection of a set of items in which order does not matter.

multiplication principle: Suppose a procedure can be broken into m successive stages, with n_1 possibilities in stage 1, n_2 possibilities in stage 2, and so on, until we have n_m possibilities in the last stage. If the number of possibilities at each stage is independent of the choices we made in the previous stages, and if the final outcomes are all distinct, then the total number of outcomes for the overall procedure is exactly $n_1 \cdot n_2 \cdots n_m$.

Let $P(n,r)$ be the number of permutations that can be built using r letters out of a total set of n letters.

Then $P(n,r) = \dfrac{n!}{(n-r)!}$.

Let $C(n,r)$ be the number of combinations that can be built using r letters out of a total set of n letters.

Then $C(n,r) = \dfrac{n!}{r!(n-r)!}$.

permutation: An arrangement of a set of items in a particular order.

Examples

Example 1

Five contestants are competing for first-, second-, and third-place trophies. How many different outcomes are possible for this competition?

Notice that the order matters here (as winning first place is quite different from winning third place).

There are 5 ways to choose the first-place winner; followed by 4 ways to choose the second-place winner (once first place is chosen, that person cannot also receive second place); and then 3 ways to choose the third-place winner.

Therefore, there are $5 \cdot 4 \cdot 3$, or 60 ways to complete this competition.

Note that this is the same as $P(5,3) = \dfrac{5!}{(5-3)!} = \dfrac{5!}{2!} = 5 \cdot 4 \cdot 3.$

Example 2

How many different ways can a little league baseball coach place kids in the 9 positions on the field if there are 11 players on his team?

The order in which the kids are placed definitely matters here. (If a child has trouble catching the ball, then making that child the catcher is not wise!) So we are looking at permutations again, and the number is going to be $P(11,9)$.

$$P(11,9) = \frac{11!}{(11-9)!} = \frac{11!}{2!} = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 19{,}958{,}400$$

Example 3

I have a set of 6 books that I want to read during my summer vacation, but I will only have time to read 4 of them. How many different ways can I choose the books to take on my trip?

The order in which I choose the books doesn't matter. I just want to choose 4 of them. So this is a combinations problem.

$$C(6,4) = \frac{6!}{4!(6-4)!} = \frac{6!}{4!2!} = \frac{6 \cdot 5}{2} = 15$$

There are 15 different collections of 4 books that I could take on my summer vacation trip.

Example 4

Your local pizza shop wants to advertise the total number of different possible 4-topping pizzas it can make, where the 4 toppings are chosen from a set of 10 possible toppings, there are 3 kinds of crust, and there are 2 kinds of sauces.

Notice that there are several stages to think about for building any one of these pizzas. So we think in stages and get our final answer using the multiplication principle.

First, choose the crust. There are 3 choices there. You are only going to use one crust, so you have $C(3, 1)$ choices, which is 3. Next, pick your sauce—so that's 2 choices. Now let's see how many different ways we can pick our 4 toppings.

Note that the order in which we pick the toppings does not matter—they are all going to be thrown on the pizza "at once," for our purposes. So this is a combinations problem. That means there are $C(10, 4)$, or 210 different ways to choose the toppings.

We use the multiplication principle to see that there are exactly $3 \cdot 2 \cdot 210$, or 1260, different 4-topping pizzas the shop can make.

Common Errors

- Confusing permutation problems with combinations problems.

Study Tips

- Take the time to ask yourself whether order matters. If it does, then the problem is most likely a permutations problem. Otherwise, it is a combinations problem.

- Don't try to just jump to one formula or another—think through these problems carefully.

Problems

Calculate the following.

1. $P(10, 10)$

2. $P(8, 4)$

3. $P(7, 5)$

4. $C(8, 4)$

5. $C(7, 5)$

Solve each of the following problems.

6. Ten contestants enter a contest. How many ways can a winner and a runner-up be determined?

7. How many different ways can 12 students line up for the bus at the end of the school day?

8. How many different 3-letter "words" can be made from the letters in the word TEACHING? (Here, a "word" doesn't necessarily need to be a valid word in English, so HNG would be one such "word.")

9. Fifteen friends go to dinner together. How many ways are there for 6 of them to order steak, 5 to order chicken, and 4 to order fish?

10. Nathan is preparing for a trip. He owns 6 pairs of pants, 8 shirts, and 3 sweater vests. How many different ways can he select 4 pairs of pants, 5 shirts, and 2 sweater vests to take on his trip?

Elementary Probability
Lesson 36

Topics in This Lesson

- Sample spaces and outcomes.

- The theoretical probability of events.

Summary

We introduce the idea of theoretical probability and discuss several examples of how to compute such probabilities.

Definitions and Formulas

sample space: The set of all possible outcomes for a particular activity or experiment.

theoretical probability: If a sample space has n equally likely outcomes and an event a occurs in m of these outcomes, then the theoretical probability of event a is denoted as $p(A)$ and determined as follows:

$$p(A) = \frac{m}{n}.$$

Examples

Example 1

What is the probability of randomly selecting a prime number from the first 20 positive integers?

The sample space is the set of numbers $\{1, 2, 3, \ldots, 20\}$. So $n = 20$. The prime numbers in the sample space are 2, 3, 5, 7, 11, 13, 17, and 19. There are 8 of these, so $m = 8$. Therefore, the probability of randomly selecting a prime number from the first 20 positive integers is 8/20, or 2/5 after we simplify the fraction.

Example 2

What is the probability of having a flush dealt to us at random from a 52-card deck?

The denominator of this probability is $C(52, 5)$ because that is the total number of possible 5-card hands that could be dealt to us from a 52-card deck. Now we need to find the numerator. That means we need to understand the number of different ways we could get a flush: a hand where all 5 cards are from the same suit (hearts, diamonds, spades, or clubs).

This can be determined using the multiplication principle. The first stage is to choose which suit we want our flush to be from. There are 4 such choices—hearts, diamonds, spades, or clubs—so stage 1 gives us 4 choices. Once we have chosen our suit, we can ignore all the cards in the other suits. That will leave us with just 13 cards in our hand. There are $C(13, 5)$ ways to choose our hand from this suit. By the multiplication principle, the numerator of our probability is $4C(13, 5)$.

So our probability is $\dfrac{4C(13,5)}{C(52,5)}$.

Example 3

When you are dialing a phone on your school campus, only the last 4 digits of the phone number have to be dialed, and those last 4 digits could be anything from 0000 to 9999. What is the probability that a 4-digit number (a number between 0000 and 9999, inclusively) is chosen at random so that the first and last digits are equal?

First, the sample space is the set of numbers from 0000 to 9999. There are 10,000 numbers in this set, so the denominator of the probability is 10,000.

The total number of outcomes in the sample space is exactly the total number of ways we can fill in 4 blanks with the digits 0 through 9.

____ ____ ____ ____

There are 10 possibilities for the first digit (anything from 0 to 9), 10 possibilities for the second digit (anything from 0 to 9 again), and 10 possibilities for the third digit. But note that there is only 1 possibility for the last digit once the first digit is chosen (since the first and last digits must be equal). Therefore, by the multiplication principle, the total number of ways to build the desired numbers is $10 \cdot 10 \cdot 10 \cdot 1$, or 1000. This means that the probability in question is 1000/10,000, or 1/10.

Common Errors

- Confusing combinations and permutations when calculating probability.

Study Tips

- Think in as organized a way as possible when counting the number of different outcomes in a problem.

Problems

1. Find the probability of choosing an even number from the set of numbers {1, 2, ... , 100}.

2. Find the probability of choosing a number that is divisible by 3 from the set of numbers {1, 2, ... , 100}.

3. Find the probability of choosing a number whose units digit is 7 from the set of numbers {1, 2, ... , 100}.

4. Assume there are 40 red marbles, 30 blue marbles, 20 white marbles, and 10 yellow marbles in a jar. Find the probability of choosing a blue marble from this jar.

5. Assume there are 40 red marbles, 30 blue marbles, 20 white marbles, and 10 yellow marbles in a jar. Find the probability of choosing a nonred marble from this jar.

6. Find the probability of randomly selecting a red ace from a standard 52-card deck.

7. Find the probability of being dealt a hand (from a standard 52-card deck of cards) that contains exactly 3 red cards and 2 black cards.

8. Find the probability of being dealt a hand (from a standard 52-card deck of cards) that contains exactly 2 kings and 2 queens (and a fifth card that is neither a king or queen).

9. Find the probability of being dealt a hand (from a standard 52-card deck of cards) that contains 4 of a kind.

10. Find the probability of being dealt a hand (from a standard 52-card deck of cards) that is a straight flush.

Formulas and Theorems

binomial theorem (special case): Let n be a positive integer. Then $(x+1)^n = C(n,0)x^n + C(n,1)x^{n-1} + C(n,2)x^{n-2} + C(n,n-1)x + C(n,n)$.

by-product of the fundamental theorem of algebra: If you include all roots (rational, real but not rational, and imaginary) and count the roots that appear multiple times, it turns out that an nth-degree polynomial equation has exactly n roots!

change of base formula: For any positive numbers M, b, and c, where $b \neq 1$ and $c \neq 1$,

$$\log_b M = \frac{\log_c M}{\log_c b}.$$

completing the square for a quadratic function: If a quadratic function is of the form $f(x) = x^2 + bx + c$, where b and c are real numbers, we must both add and subtract the quantity $(b/2)$, or $b^2/4$, on the polynomial side.

Descartes' rule of signs: Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$; let C_p equal the number of sign changes between the coefficients (when they are read left-to-right) of $p(x)$; and let C_n equal the number of sign changes between the coefficients of $p(-x)$. Then the number of positive real zeros of $p(x)$ is C_p, $C_p - 2$, $C_p - 4$, ... , down to either 1 or 0 (depending on whether C_p is odd or even); and the number of negative real zeros of $p(x)$ is C_n, $C_n - 2$, $C_n - 4$, ... , down to either 1 or 0 (depending on whether C_n is odd or even).

factor theorem: The expression $x - a$ is a linear factor of a polynomial function if and only if the value a is a zero of the polynomial function.

factoring using differences of 2 cubes:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

fundamental theorem of algebra: If $p(x)$ is a polynomial of degree at least 1, then the equation $p(x) = 0$ has at least one root.

laws of exponents:

1. $x^a \cdot x^b = x^{a+b}$

2. $\dfrac{x^a}{x^b} = x^{a-b}$

3. $(x^a)^b = x^{ab}$

logarithm: If $y = b^x$, then the logarithm to the base b of a positive number y is denoted by $\log_b y$ and is defined by $\log_b y = x$.

multiplication principle: Suppose a procedure can be broken into m successive stages, with n_1 possibilities in stage 1, n_2 possibilities in stage 2, and so on, until we have n_m possibilities in the last stage. If the number of possibilities at each stage is independent of the choices we made in the previous stages, and if the final outcomes are all distinct, then the total number of outcomes for the overall procedure is exactly $n_1 \cdot n_2 \cdots n_m$.

Let $P(n,r)$ be the number of permutations that can be built using r letters out of a total set of n letters.

Then $P(n,r) = \dfrac{n!}{(n-r)!}$.

Let $C(n,r)$ be the number of combinations that can be built using r letters out of a total set of n letters.

Then $C(n,r) = \dfrac{n!}{r!(n-r)!}$.

Pascal's triangle, formula for elements of: Let $C(n,r)$ be the element in Pascal's triangle in the nth row and r^{th} column.

For any $n \geq 0$ and $r \geq 0$, $C(n,r) = \dfrac{n!}{r!(n-r)!}$.

pert formula: $A = Pe^{rt}$, where A is the amount of money in one's account at any time t, P is the amount of principal (the amount you put in initially, or when $t = 0$), r is the interest rate, and t is time (in years). This formula only applies when the interest on the account is compounded continuously.

properties of logarithms: Let M, N, and b be positive numbers with $b \neq 1$, and let x be any real number.

1. $\log_b(MN) = \log_b M + \log_b N$

2. $\log_b\left(\dfrac{M}{N}\right) = \log_b M - \log_b N$

3. $\log_b\left(M^x\right) = x\log_b M$

quadratic formula: For a quadratic equation of the form $ax^2 + bx + c = 0$, the solutions can be found by using the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

rational roots theorem: The only possible rational roots of the polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ are of the form p/q, where p/q is already in reduced form, p is a factor of a_0, and q is a factor of a_n.

standard form of the equation of a circle: The standard form for the equation of a circle centered at the origin is given by $x^2 + y^2 = r^2$, where r is the radius of the circle.

standard form of the equation of a hyperbola: The standard form for a hyperbola looks like $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ or $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$, where a and b are real numbers.

standard form of the equation of an ellipse: The standard form of the equation of an ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ or $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$, where a and b are nonzero numbers and $a > b$.

vertical line test: A test used to determine whether a graph represents a function. If you can draw a vertical line that crosses through at least 2 points on a graph, then the graph does not represent a function.

vertices and asymptotes of a hyperbola:

In the case of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, the vertices are at the points $(a, 0)$ and $(-a, 0)$, and the asymptotes are given by $y = \dfrac{b}{a} x$ and $y = -\dfrac{b}{a} x$.

In the case of $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$, the vertices are at the points $(0, a)$ and $(0, -a)$, and the asymptotes are given by $y = \dfrac{a}{b} x$ and $y = -\dfrac{a}{b} x$.

Solutions

1.
$$14 \times (8 - 5) + 2^5$$
$$= 14 \times 3 + 2^5$$
$$= 14 \times 3 + 32$$
$$= 42 + 32$$
$$= 74$$

2.
$$(-5 - 3)^2 + 100$$
$$= (-8)^2 + 100$$
$$= 64 + 100$$
$$= 164$$

3.
$$17 + 3 \cdot 5 - 4 \div 2 + 19$$
$$= 17 + 15 - 2 + 19$$
$$= 32 - 2 + 19$$
$$= 30 + 19$$
$$= 49$$

4.
$$5 + 3^2 - 6 \cdot 7$$
$$= 5 + 9 - 6 \cdot 7$$
$$= 5 + 9 - 42$$
$$= 14 - 42$$
$$= -28$$

5.
$$1 + 2 - 3 \cdot 4 \div (5 - 2)$$
$$= 1 + 2 - 3 \cdot 4 \div 3$$
$$= 1 + 2 - 12 \div 3$$
$$= 1 + 2 - 4$$
$$= 3 - 4$$
$$= -1$$

6. $(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$

7. $((-2)^3)^3 = (-2)^{3 \cdot 3} = (-2)^9 = -2^9 = -512$

8. $\quad 4^7 \div 4^2 = 4^{(7-2)} = 4^5 = 1024$

9. $\quad 5^2 \div 5^{-4} = 5^{(2-(-4))} = 5^{(2+4)} = 5^6$

10. $\quad 2^3 \cdot 2^{-6} \cdot 2^8 = 2^{3-6+8} = 2^5 = 32$

Lesson 2

1.
$$\begin{aligned} x + 5 &= -5x + 5 \\ x + 5 - 5 &= -5x + 5 - 5 \\ x &= -5x \\ x + 5x &= -5x + 5x \\ 6x &= 0 \\ x &= 0 \end{aligned}$$

The solution is $x = 0$.

2.
$$\begin{aligned} x - 1 &= 6x + 2x - 8 \\ x - 1 &= 8x - 8 \\ x &= 8x - 7 \\ -7x &= -7 \\ x &= 1 \end{aligned}$$

The solution is $x = 1$.

3.
$$\begin{aligned} 5x - 14 &= 8x + 4 \\ 5x &= 8x + 18 \\ -3x &= 18 \\ x &= -6 \end{aligned}$$

The solution is $x = -6$.

4.
$$-4(-6x-3) = 12$$
$$24x + 12 = 12$$
$$24x = 0$$
$$x = 0$$

The solution is $x = 0$.

5.
$$-(7-4x) = 9$$
$$-7 + 4x = 9$$
$$4x = 16$$
$$x = 4$$

The solution is $x = 4$.

6.
$$-18 - 6x = 6(1 + 3x)$$
$$-18 - 6x = 6 + 18x$$
$$-6x = 24 + 18x$$
$$-24x = 24$$
$$x = -1$$

The solution is $x = -1$.

7.
$$2(4x-3) - 8 = 2x + 4$$
$$8x - 6 - 8 = 2x + 4$$
$$8x - 14 = 2x + 4$$
$$8x = 2x + 18$$
$$6x = 18$$
$$x = 3$$

The solution is $x = 3$.

8.
$$3x - 5 = -8(5x + 6)$$
$$3x - 5 = -40x - 48$$
$$43x - 5 = -48$$
$$43x = -43$$
$$x = -1$$

The solution is $x = -1$.

9.

$$-3(4x+3)+4(6x+1)=19$$
$$-12x-9+24x+4=19$$
$$12x-5=19$$
$$12x=24$$
$$x=2$$

The solution is $x=2$.

10.

$$28x-22=-4(-7x+1)$$
$$28x-22=28x-4$$
$$-22=-4$$

There is no solution.

Lesson 3

1. $\left|-x+7\right|=19$

$-x+7=19$ or $-x+7=-19$

$$-x+7=19$$
$$-x=12$$
$$x=-12$$

$$-x+7=-19$$
$$-x=-26$$
$$x=26$$

Check it.

$$\left|-x+7\right|=19$$
$$\left|-(-12)+7\right|=19$$
$$\left|12+7\right|=19$$
$$\left|19\right|=19$$
$$19=19 \quad \text{Correct. So } x=-12 \text{ is a solution.}$$

$$\left|-x+7\right|=19$$
$$\left|-26+7\right|=19$$
$$\left|-19\right|=19$$
$$19=19 \quad \text{Correct. So } x=26 \text{ is a solution.}$$

2. $\left| 2x+3 \right| = 7$

$2x + 3 = 7 \ \ \text{or} \ \ 2x + 3 = -7$

$2x + 3 = 7$
$2x = 4$
$x = 2$

$2x + 3 = -7$
$2x = -10$
$x = -5$

Check it.

$$\left| 2x+3 \right| = 7$$
$$\left| 2(2)+3 \right| = 7$$
$$\left| 4+3 \right| = 7$$
$$\left| 7 \right| = 7$$
$$7 = 7 \ \ \text{Correct. So } x = 2 \text{ is a solution.}$$

$$\left| 2x+3 \right| = 7$$
$$\left| 2(-5)+3 \right| = 7$$
$$\left| -10+3 \right| = 7$$
$$\left| -7 \right| = 7$$
$$7 = 7 \ \ \text{Correct. So } x = -5 \text{ is a solution.}$$

3. $-2\left| x+2 \right| + 12 = 0$
$-2\left| x+2 \right| = -12$
$\left| x+2 \right| = 6$

$x + 2 = 6 \ or \ x + 2 = -6$

$x + 2 = 6$
$x = 4$

$x + 2 = -6$
$x = -8$

Check it.

$$-2\,\big|\,x+2\,\big|+12=0$$
$$-2\,\big|\,4+2\,\big|+12=0$$
$$-2\,\big|\,6\,\big|+12=0$$
$$-2\big(6\big)+12=0$$
$$-12+12=0$$
$$0=0 \quad \text{Correct. So } x=4 \text{ is a solution.}$$

$$-2\,\big|\,x+2\,\big|+12=0$$
$$-2\,\big|\,-8+2\,\big|+12=0$$
$$-2\,\big|\,-6\,\big|+12=0$$
$$-2\big(6\big)+12=0$$
$$-12+12=0$$
$$0=0 \quad \text{Correct. So } x=-8 \text{ is a solution.}$$

4. $\quad \big|\,3x\,+\,2\,\big|+8=0$

$\qquad \big|\,3x\,+\,2\,\big|=-8$

But an absolute value can never equal −8. So there are no solutions to the original equation.

5. $\quad \big|\,2x+12\,\big|=7x-3$

$\qquad 2x+12=7x-3 \ \ \text{or} \ \ 2x+12=-\big(7x-3\big)$

$\qquad 2x+12=7x-3$
$\qquad -5x+12=-3$
$\qquad\quad -5x=-15$
$\qquad\qquad x=3$

$\qquad 2x+12=-\big(7x-3\big)$
$\qquad 2x+12=-7x+3$
$\qquad 9x+12=3$
$\qquad\quad 9x=-9$
$\qquad\qquad x=-1$

Check it.

$$|2x+12| = 7x-3$$
$$|2(3)+12| = 7(3)-3$$
$$|6+12| = 21-3$$
$$|18| = 18$$
$$18 = 18$$

This is correct. So $x = 3$ is a solution.

$$|2x+12| = 7x-3$$
$$|2(-1)+12| = 7(-1)-3$$
$$|-2+12| = -7-3$$
$$|10| = -10$$
$$10 = -10$$

This is false. So $x = -1$ is not a solution.

Therefore, the only solution of the original equation is $x = 3$.

6. $|x+6| = 3x$
        $x+6 = 3x$ or $x+6 = -3x$

        $x+6 = 3x$
      $-2x+6 = 0$
          $-2x = -6$
            $x = 3$

        $x+6 = -3x$
      $4x+6 = 0$
          $4x = -6$
            $x = -6/4 = -3/2$

Check it.

$$|x+6| = 3x$$
$$|3+6| = 3(3)$$
$$|9| = 9$$
$$9 = 9$$

This is correct. So $x = 3$ is a solution.

$$|x+6| = 3x$$
$$|-3/2+6| = 3(-3/2)$$
$$|9/2| = -9/2$$
$$9/2 = -9/2$$

This is false. So $x = -3/2$ is not a solution.

Therefore, the only solution of the original equation is $x = 3$.

7.
$$3|x+7| = 36$$
$$|x+7| = 12$$
$$x+7 = 12 \text{ or } x+7 = -12$$

$$x+7 = 12$$
$$x = 5$$

$$x+7 = -12$$
$$x = -19$$

Check it.

$$3|x+7| = 36$$
$$3|5+7| = 36$$
$$3|12| = 36$$
$$3(12) = 36$$
$$36 = 36$$

This is correct. So $x = 5$ is a solution.

$$3|x+7| = 36$$
$$3|-19+7| = 36$$
$$3|-12| = 36$$
$$3(12) = 36$$
$$36 = 36$$

This is correct. So $x = -19$ is a solution.

8. $\left| (1/2)x + 2 \right| = 8$

$(1/2)x + 2 = 8$ or $(1/2)x + 2 = -8$
$x + 4 = 16$ or $x + 4 = -16$ (by multiplying through by 2 in both equations)

$x + 4 = 16$
$x = 12$

$x + 4 = -16$
$x = -20$

Check it.

$\left| (1/2)x + 2 \right| = 8$
$\left| (1/2)(12) + 2 \right| = 8$
$\left| 6 + 2 \right| = 8$
$\left| 8 \right| = 8$
$8 = 8$

This is correct. So $x = 12$ is a solution.

$\left| (1/2)x + 2 \right| = 8$
$\left| (1/2)(-20) + 2 \right| = 8$
$\left| -10 + 2 \right| = 8$
$\left| -8 \right| = 8$
$8 = 8$

This is correct. So $x = -20$ is a solution.

9. $\left| 2x + 10 \right| = 0$
$2x + 10 = 0$
$2x = -10$
$x = -5$

Check it.

$\left| 2x + 10 \right| = 0$
$\left| 2(-5) + 10 \right| = 0$
$\left| -10 + 10 \right| = 0$
$\left| 0 \right| = 0$
$0 = 0$

This is correct. So $x = -5$ is a solution.

10. $\left| 2x + 7 \right| = x - 4$

$2x + 7 = x - 4$ or $2x + 7 = -\left(x - 4\right)$

$2x + 7 = x - 4$
$x + 7 = -4$
$x = -11$

$2x + 7 = -\left(x - 4\right)$
$2x + 7 = -x + 4$
$3x + 7 = 4$
$3x = -3$
$x = -1$

Check it.

$\left| 2x + 7 \right| = x - 4$
$\left| 2(-11) + 7 \right| = -11 - 4$
$\left| -22\ +\ 7 \right| = -15$
$\left| -15 \right| = -15$
$15 = -15$

This is false. So x -11 is not a solution.

$\left| 2x + 7 \right| = x - 4$
$\left| 2(-1) + 7 \right| = -1 - 4$
$\left| -2 + 7 \right| = -5$
$\left| 5 \right| = -5$
$5 = -5$

This is false. So $x = -1$ is not a solution.

Therefore, the original equation has no solutions.

1.   The line must have slope 3 and pass through the point (4, 2).

We can use the point-slope formula here since we are given the slope and a point on the line. So we write $y - 2 = 3(x - 4)$. This is certainly a valid final answer. If we wish to write this equation in slope-intercept form, we can do that as well.

$$y - 2 = 3(x - 4)$$
$$y - 2 = 3x - 12$$
$$y = 3x - 10$$

This would also be a valid final answer.

2.   The line must be parallel to the line $y = 7x + 3$ and have y-intercept 4.

If the line must be parallel to the line $y = 7x + 3$, then it must have slope 7. So we know the slope of the new line and its y-intercept. Using the slope-intercept form of the equation of the line, we have $y = 7x + 4$.

3.   The line must be perpendicular to the line $y = x/2 + 5$ and pass through the origin.

The line $y = x/2 + 5$ has slope 1/2. So our new line must have slope -2 (since -2 and 1/2 are negative reciprocals of one another). Since this new line is to pass through the origin, we know its y-intercept is 0. Therefore, the equation of this new line is $y = -2x + 0$, or simply $y = -2x$.

4.   The line must pass through the points (1, 2) and (4, 8).

From these 2 points, we can determine the slope of our line. It is given by $\dfrac{8-2}{4-1} = \dfrac{6}{3} = 2$. We can now use the

point-slope formula with the slope 2 and either of the 2 points that were given to us to write the equation of the line. It is $y - 2 = 2(x - 1)$. When written in slope-intercept form, we have the below.

$$y - 2 = 2(x - 1)$$
$$y - 2 = 2x - 2$$
$$y = 2x$$

5. $x + y = 9; x - y = 2$

We need to determine the slopes of the 2 lines that were given to us.

$$x + y = 9$$
$$y = -x + 9$$

This line has slope -1.

$$x - y = 2$$
$$-y = -x + 2$$
$$y = x - 2$$

This line has slope 1.

Since these 2 slopes are different, we know that these 2 lines are not parallel.

6. $y = 4x + 3; 8x - 2y = 24$

The slope of the first line is 4. We now need to determine the slope of the second line.

$$8x - 2y = 24$$
$$-2y = -8x + 24$$
$$y = 4x - 12$$

So the slope of the second line is also 4. Note that the 2 lines have different y-intercepts (one is at 3, the other is at -12). Therefore, these 2 lines are parallel.

7. $4x + 2y = 10; 10x + 5y = 25$

We determine the slopes of each of the lines.

$$4x + 2y = 10$$
$$2y = -4x + 10$$
$$y = -2x + 5$$

$$10x + 5y = 25$$
$$5y = -10x + 25$$
$$y = -2x + 5$$

Not only do we see that the 2 lines have the same slope (-2), but we also see that they are actually the same line! Given that they have the same slope and the same y-intercept, we conclude that they are not parallel (they are simply the same line).

8. $y = 6x + 9$; $x + 6y = 14$

To determine whether these 2 lines are perpendicular, we must determine their slopes and see if they are negative reciprocals of one another. The slope of the first line is 6. To determine the slope of the second line, we rewrite it in slope-intercept form.

$$
\begin{aligned}
x + 6y &= 14 \\
6y &= -x + 14 \\
y &= -\frac{1}{6}x + \frac{14}{6} \\
y &= -\frac{1}{6}x + \frac{7}{3}
\end{aligned}
$$

So the slope of the second line is $-\dfrac{1}{6}$, which is the negative reciprocal of 6, the other line's slope. Therefore, the 2 lines are perpendicular.

9. $y = 3x - 2$; $y = x/3$

The slope of the first line is 3, while the slope of the second line is 1/3. These are reciprocals, but they are not negative reciprocals of one another. Therefore, these lines are not perpendicular to one another.

10. $y = 7$; $x = -2$

The first line is a horizontal line (with y-intercept 7). The second line is a vertical line (with x-intercept -2). Therefore, these 2 lines are perpendicular (even though we cannot compute the slope of a vertical line) because they intersect at right angles.

1. $y = -2x + 3$

The graph of this equation is a line with slope -2 and y-intercept 3. So we know, for example, that the graph goes through the points $(0, 3)$ and $(1, 1)$. Hence, the graph looks like the following.
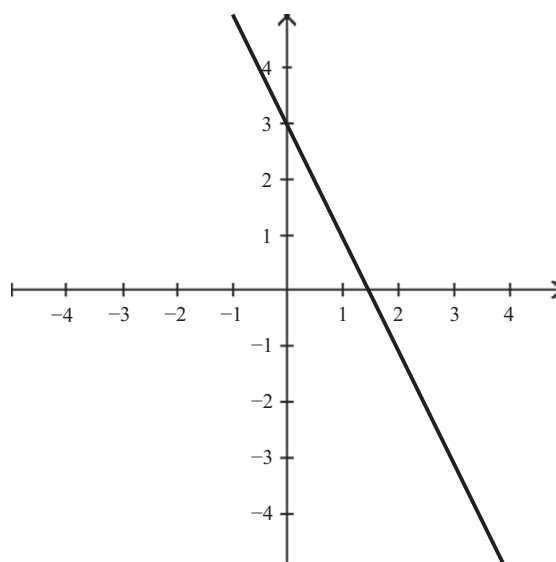
2. $y = -3$

This is the equation of the horizontal line passing through the y-intercept $(0, -3)$. Thus, its graph is the following.
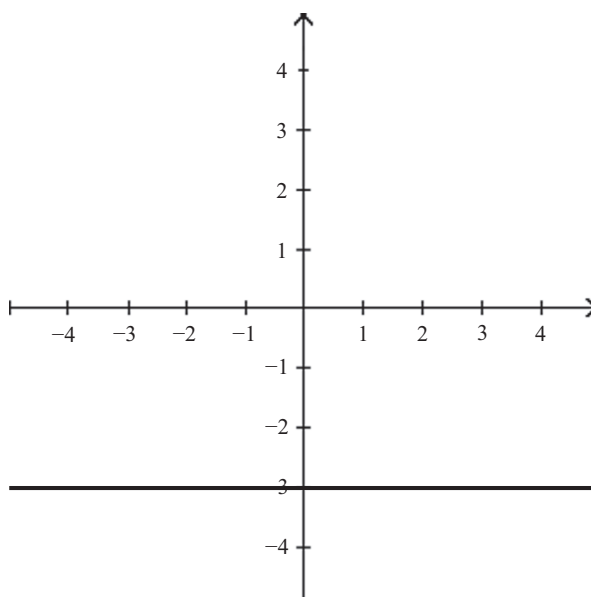
3. $3x + y = 4$

We begin by rewriting the equation in slope-intercept form.

$$3x + y = 4$$
$$y = -3x + 4$$

Now we see that the slope of the line is −3 and the y-intercept is 4. So, we know the line goes through (0, 4), and we know it also goes through (1, 1). Therefore, the graph is as follows.
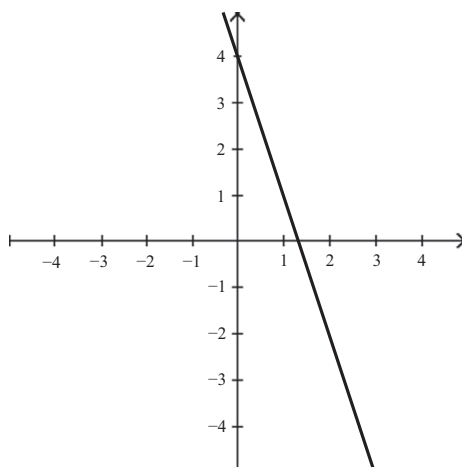
4. $6x - 3y = 18$

We start by rewriting the equation in slope-intercept form:

$$6x - 3y = 18$$
$$-3y = -6x + 18$$
$$y = 2x - 6$$

So the graph looks like the following.
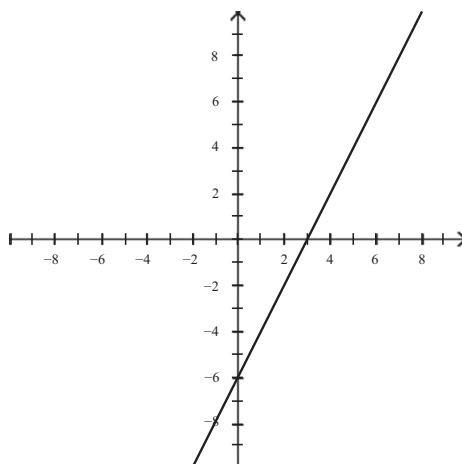
5. $2(3x + y) = 14$

We rewrite the equation in slope-intercept form:

$$2(3x + y) = 14$$
$$6x + 2y = 14$$
$$2y = -6x + 14$$
$$y = -3x + 7$$

So the graph looks like the following.
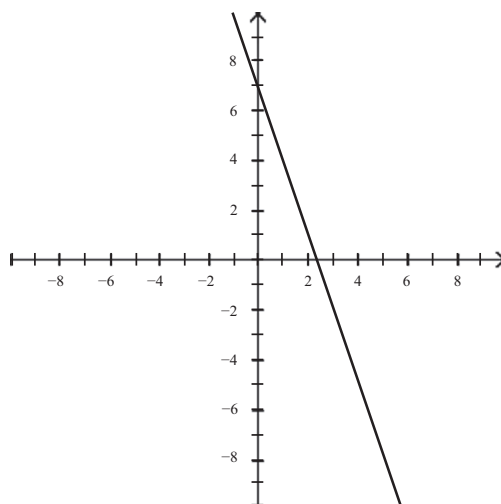
6. $y = |x| + 2$

We see here that we have added 2 to the whole function, not just to the x. So the graph of this function is the same as the graph of $y = |x|$ but shifted up 2 units. Therefore, it looks like the following.
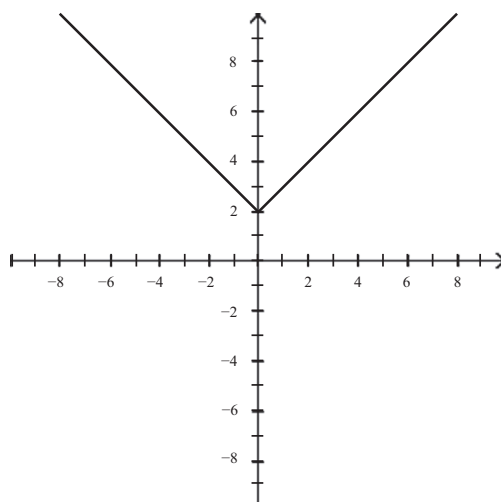
7. $y = |x - 3|$

Here we have subtracted 3 from x in the function. This means we must move the graph of $y = |x|$ to the right by 3 units in order to get the graph we want.
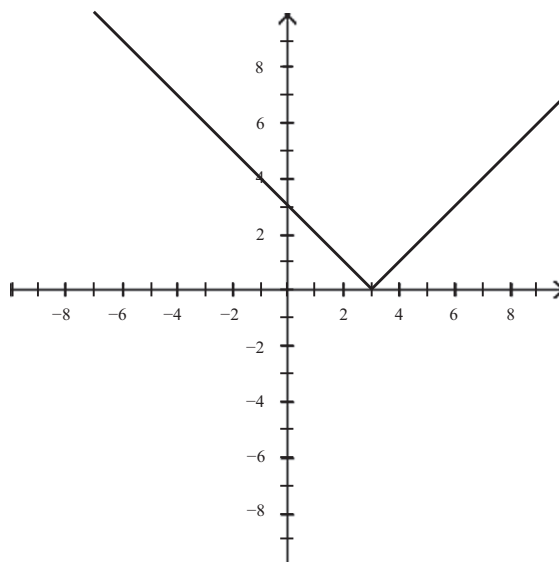
8. $y = |x + 1| + 4$
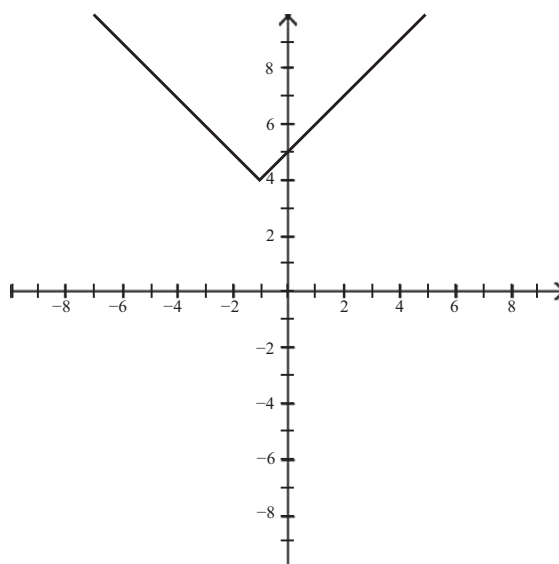
We have now performed 2 different changes to the original function. We added 1 to the x, which means we must move the original graph to the left by 1 unit. We also added 4 to the entire function, which means we must shift the original graph up by 4 units. Once we perform both of these shifts, we will have our desired graph.

9. $y = -|x|$

In the lesson, we learned that multiplying the whole function by −1 flips the graph of the original function over the *x*-axis. So in our case, the "V" shape of the graph of $y = |x|$ will be flipped over to an upside-down "V" (which is still touching the origin). So the graph of $y = -|x|$ is as follows.
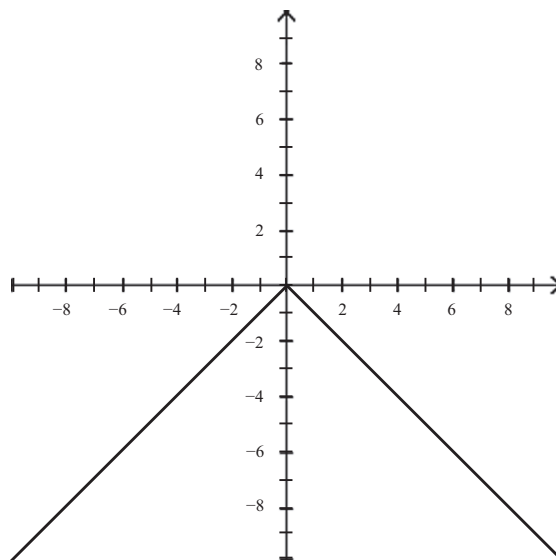
10. $y = |2x|$

We have not seen yet what can be done to the graph when we multiply *x* by a number like 2. So I suggest just plotting several points and connecting the dots. After doing so, the graph is as follows.
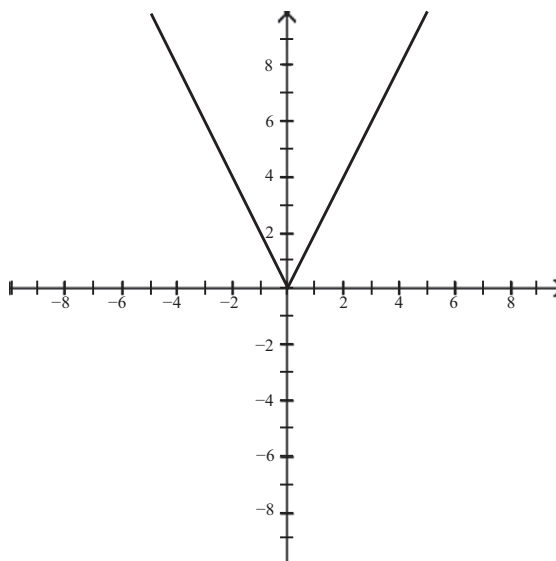
1. $f(x) = -3x + 5$

The domain of this function is the set of all real numbers, because we may plug any real number we wish into this function. So we may denote this domain by $(-\infty, \infty)$.

2. $f(x) = \sqrt{2x+4}$

For this function, we must have $2x + 4 \geq 0$ or $2x \geq -4$ or $x \geq -2$. So the domain is the interval $[-2, \infty)$.

3. $f(x) = \dfrac{x+2}{x-3}$

We need to avoid division by zero, which means that we cannot allow $x = 3$ in the domain of this function. However, all other real numbers are allowed in the domain. So the domain is $(-\infty, 3) \cup (3, \infty)$.

4. $f(x) = \dfrac{1}{\sqrt{x-5}}$

In this problem, we need to avoid division by zero as well as negative numbers within the square root symbol. That means we must only allow values of x such that $y = x^3 - x$ This means the domain is the set of all real numbers $x > 5$, which can be written as $(5, \infty)$.

5. $f(x) = 2x + 1;\ g(x) = 5x - 1$

$f(g(x)) = f(5x - 1) = 2(5x - 1) + 1 = 10x - 2 + 1 = 10x - 1$

$g(f(x)) = g(2x + 1) = 5(2x + 1) - 1 = 10x + 5 - 1 = 10x + 4$

6. $f(x) = x^2;\ g(x) = 2x$

$f(g(x)) = f(2x) = (2x)^2 = 4x^2$

$g(f(x)) = g(x^2) = 2x^2$

7. $f(x) = -x + 1; \; g(x) = x - 1$

$f(g(x)) = f(x - 1) = -(x - 1) + 1 = -x + 1 + 1 = -x + 2$

$g(f(x)) = g(-x + 1) = (-x + 1) - 1 = -x$

8.   This graph satisfies the vertical line test; there is no vertical line that can be drawn through this graph that goes through at least 2 points on the graph. So this graph represents a function.

9.   This graph does not satisfy the vertical line test; we can draw a vertical line through the graph that touches the graph at more than 1 point. So this graph does not represent a function.

10.   This graph does not satisfy the vertical line test; we can draw a vertical line through the graph that touches the graph at more than 1 point. So this graph does not represent a function.

Lesson 7

1. $y = 5x - 6$
$y = -5x + 4$

The graphs of the equations are as follows.
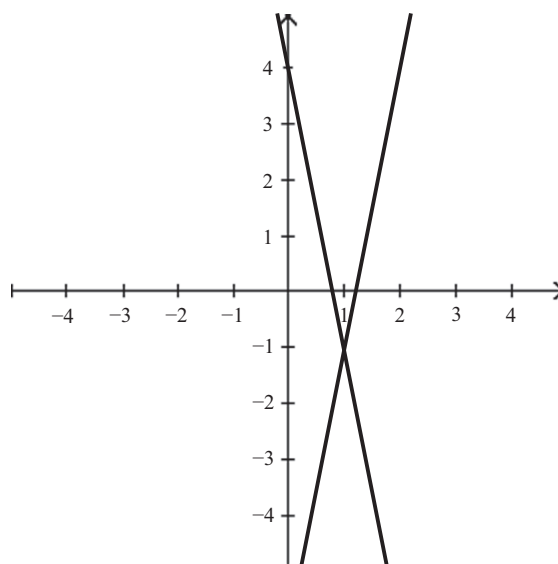
The solution appears to be at the point $(1, -1)$. Let's check this.

$$y = 5x - 6$$
$$-1 = 5(1) - 6$$
$$-1 = 5 - 6$$
$$-1 = -1$$

This is true.

$$y = -5x + 4$$
$$-1 = -5(1) + 4$$
$$-1 = -5 + 4$$
$$-1 = -1$$

This is also true.

2. $-4x + 3y = 9$
$2x + 3y = -9$

We begin by rewriting the original equations in slope-intercept form. This will make the graphing process easier.

$$2x + 3y = -9$$
$$3y = -2x - 9$$
$$y = -\frac{2}{3}x - 3$$

The graphs of these equations are as follows.
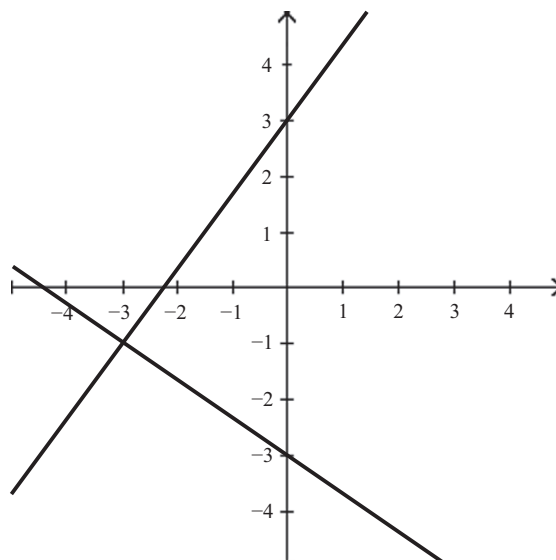
The solution appears to be at the point $(-3, -1)$. Let's check.

$$-4x + 3y = 9$$
$$-4(-3) + 3(-1) = 9$$
$$12 - 3 = 9$$
$$9 = 9$$

This is true.

$$2x + 3y = -9$$
$$2(-3) + 3(-1) = -9$$
$$-6 - 3 = -9$$
$$-9 = -9$$

This is also true.

3. $y = \dfrac{2}{3}x + 1$

$y = \dfrac{2}{3}x - 4$

The graphs of the equations are as follows.
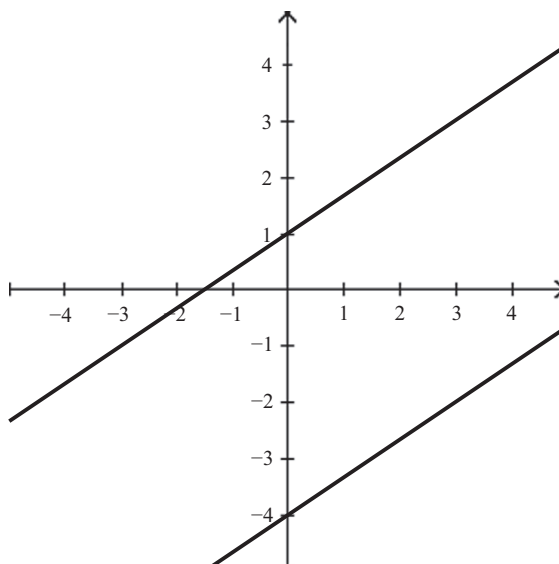
These are parallel lines, so they do not intersect. Therefore, there are no solutions for this system of linear equations. The system is inconsistent.

4. $y = -x - 2$
$-x + 3y = 6$

One of these equations, $y = -x - 2$, is already in slope-intercept form. We place the other equation in slope-intercept form in order to graph it quickly.

$$-x + 3y = 6$$
$$3y = x + 6$$
$$y = \frac{1}{3}x + 2$$

The graphs of these equations are as follows.
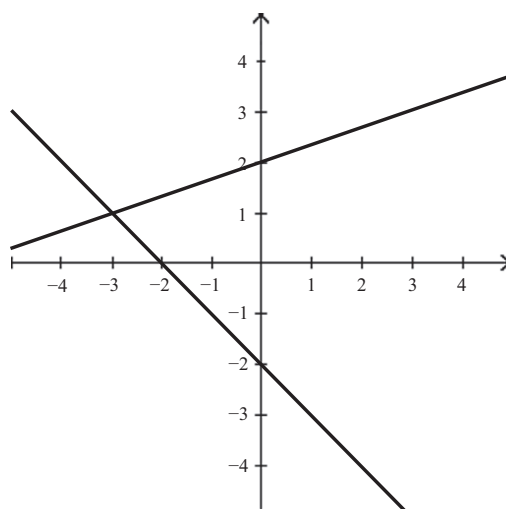
The solution of this system appears to be $(-3, 1)$, or $x = -3$, $y = 1$. Let's check.

$$y = -x - 2$$
$$1 = -(-3) - 2$$
$$1 = 3 - 2$$
$$1 = 1$$

This is true.

$$-x + 3y = 6$$
$$-(-3) + 3(1) = 6$$
$$3 + 3 = 6$$
$$6 = 6$$

This is also true.

Solutions

5. $3x + 2y = -8$
$-x + 2y = 8$

We rewrite the equations in slope-intercept form.

$$3x + 2y = -8$$
$$2y = -3x - 8$$
$$y = -\frac{3}{2}x - 4$$

$$-x + 2y = 8$$
$$2y = x + 8$$
$$y = \frac{1}{2}x + 4$$

The graphs of these equations are as follows.
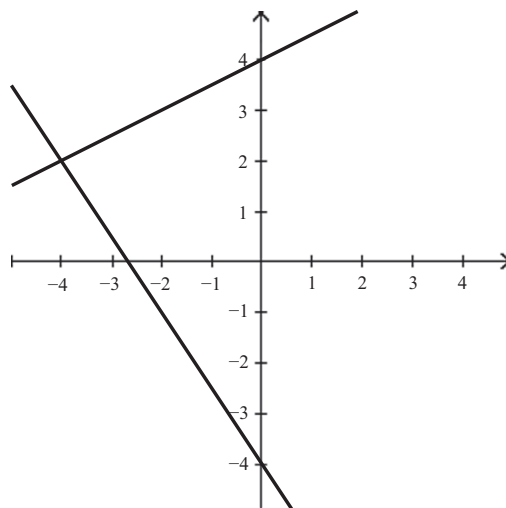
The solution of this system appears to be at $(-4, 2)$. Let's check.

$$3x + 2y = -8$$
$$3(-4) + 2(2) = -8$$
$$-12 + 4 = -8$$
$$-8 = -8$$

This is true.

$$-x + 2y = 8$$
$$-(-4) + 2(2) = 8$$
$$4 + 4 = 8$$
$$8 = 8$$

This is also true.

6. $y = 2x - 1$
        $y = -x + 2$

The graphs of the equations are as follows.
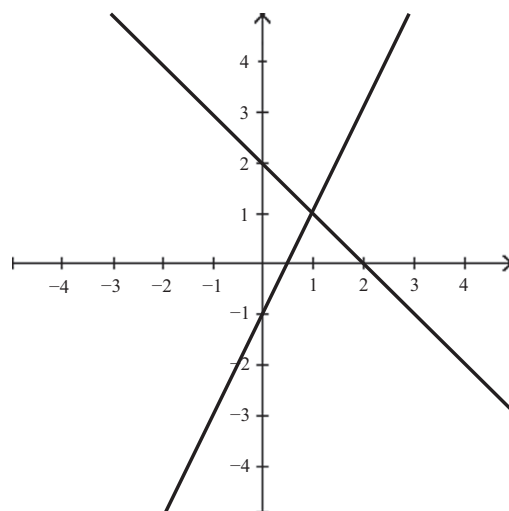
The solution of the system appears to be at (1, 1). Let's check.

$$y = 2x - 1$$
$$1 = 2(1) - 1$$
$$1 = 2 - 1$$
$$1 = 1$$

This is true.

$$y = -x + 2$$
$$1 = -1 + 2$$
$$1 = 1$$

This is also true.

7. $y = \dfrac{4}{9}x + 5$

$y = \dfrac{2}{7}x + 5$

The graphs of the equations are as follows.
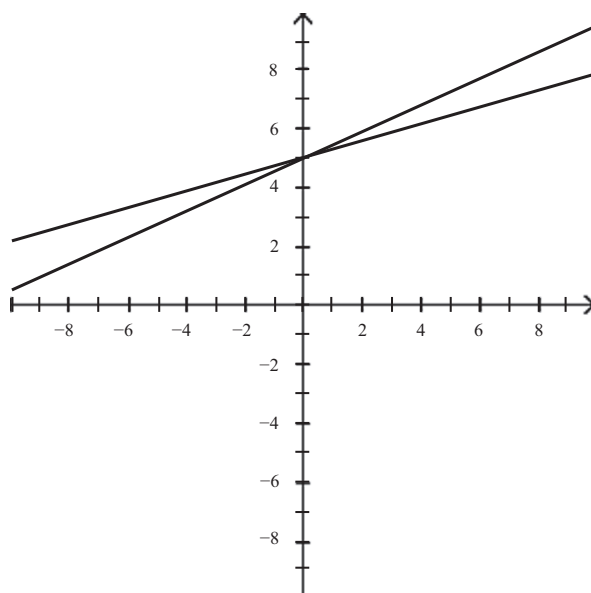
The solution appears to be at (0, 5). Let's check.

$y = \dfrac{4}{9}x + 5$

$5 = \dfrac{4}{9}(0) + 5$

$5 = 0 + 5$

$5 = 5$

$y = \dfrac{2}{7}x + 5$

$5 = \dfrac{2}{7}(0) + 5$

$5 = 0 + 5$

$5 = 5$

8. $y = -\dfrac{1}{2}x + 1$

   $y = -\dfrac{3}{2}x - 3$

The graphs of these equations are as follows.
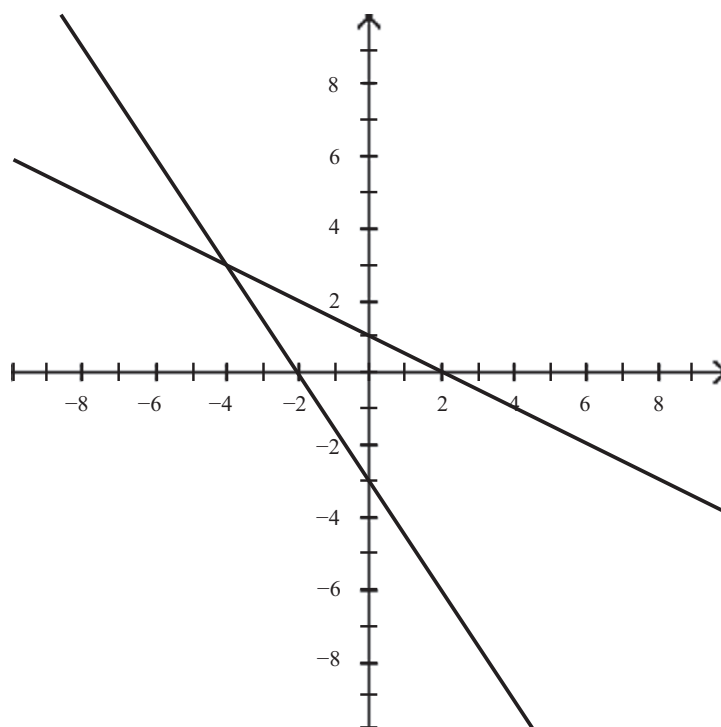
The solution of this system appears to be at $(-4, 3)$. Let's check:

$y = -\dfrac{1}{2}x + 1$

$3 = -\dfrac{1}{2}(-4) + 1$

$3 = 2 + 1$

$3 = 3$

$y = -\dfrac{3}{2}x - 3$

$3 = -\dfrac{3}{2}(-4) - 3$

$3 = 6 - 3$

$3 = 3$

9. $x - y = 4$
 $x + y = -8$

We convert each equation to slope-intercept form before graphing.

$x - y = 4$
$-y = -x + 4$
$y = x - 4$

$x + y = -8$
$y = -x - 8$
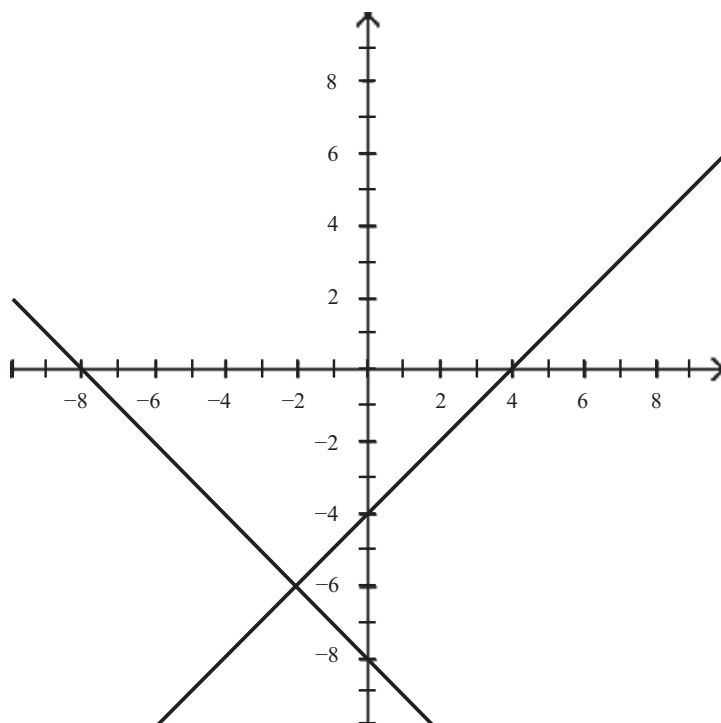
It appears that the solution is at (−2, −6). Let's check.

$x - y = 4$
$-2 - (-6) = 4$
$-2 + 6 = 4$
$4 = 4$

$x + y = -8$
$-2 + (-6) = -8$
$-2 - 6 = -8$
$-8 = -8$

10. $y = 3x + 4$
           $y = -2x + 14$

The graphs of the equations are as follows.
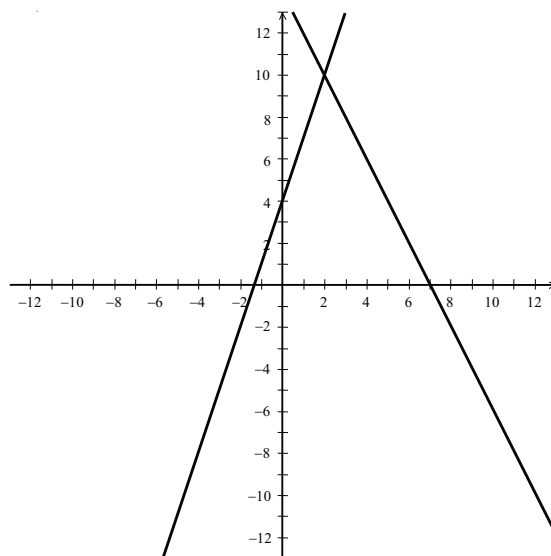
The solution appears to be at (2, 10). Let's check.

$$y = 3x + 4$$
$$10 = 3(2) + 4$$
$$10 = 6 + 4$$
$$10 = 10$$

$$y = -2x + 14$$
$$10 = -2(2) + 14$$
$$10 = -4 + 14$$
$$10 = 10$$

Lesson 8

1. $y = 2x + 7$
          $y = 4x - 19$

Substituting $2x + 7$ for y in the second equation gives $2x + 7 = 4x - 19$. Now we solve this equation for x.

$$2x + 7 = 4x - 19$$
$$7 = 2x - 19$$
$$26 = 2x$$
$$x = 13$$

We now plug this value of x into the first of the original equations to get y.

$$y = 2x + 7$$
$$y = 2(13) + 7$$
$$y = 26 + 7$$
$$y = 33$$

So the solution of this system is $x = 13$, $y = 33$. Let's check.

$$33 = 2(13) + 7$$
$$33 = 26 + 7$$
$$33 = 33$$

$$y = 4x - 19$$
$$33 = 4(13) - 19$$
$$33 = 52 - 19$$
$$33 = 33$$

2. $y = -5x + 3$
$x + y = 27$

Substituting $-5x + 3$ for y in the second equation gives $x + (-5x + 3) = 27$. Now we solve this equation for x.

$$x + (-5x + 3) = 27$$
$$x - 5x + 3 = 27$$
$$-4x + 3 = 27$$
$$-4x = 24$$
$$x = -6$$

We now plug this value of x into the first of the original equations to get y.

$$y = -5x + 3$$
$$y = -5(-6) + 3$$
$$y = 30 + 3$$
$$y = 33$$

So the solution of the system is $x = -6$, $y = 33$. Let's check.

$$y = -5x + 3$$
$$33 = -5(-6) + 3$$
$$33 = 30 + 3$$
$$33 = 33$$

$$x + y = 27$$
$$-6 + 33 = 27$$
$$27 = 27$$

3. $y = 5x - 7$
$3x + 2y = 12$

Substituting $5x - 7$ for y in the second equation gives $3x + 2(5x - 7) = 12$. Now we solve this equation for x.

$$3x + 2(5x - 7) = 12$$
$$3x + 10x - 14 = 12$$
$$13x - 14 = 12$$
$$13x = 26$$
$$x = 2$$

We now plug this value of x into the first of the original equations to get y.

$$y = 5x - 7$$
$$y = 5(2) - 7$$
$$y = 10 - 7$$
$$y = 3$$

So the solution of the system of equations is $x = 2$, $y = 3$. Let's check.

$$y = 5x - 7$$
$$3 = 5(2) - 7$$
$$3 = 10 - 7$$
$$3 = 3$$

$$3x + 2y = 12$$
$$3(2) + 2(3) = 12$$
$$6 + 6 = 12$$
$$12 = 12$$

4. $-5x + y = -2$
$3x - 6y = 12$

We rewrite the first equation as $y = 5x - 2$. Then we substitute $5x - 2$ in for y in the second equation.

$$3x - 6(5x - 2) = 12$$
$$3x - 30x + 12 = 12$$
$$-27x + 12 = 12$$
$$-27x = 0$$
$$x = 0$$

Using the first of the original equations, we see that if $x = 0$, then $y = -2$. So this is the solution of the system. Let's check it.

$$-5x + y = -2$$
$$-5(0) + (-2) = -2$$
$$0 - 2 = -2$$
$$-2 = -2$$

$$3x - 6y = 12$$
$$3(0) - 6(-2) = 12$$
$$0 + 12 = 12$$
$$12 = 12$$

5. $-3x + 3y = 4$
$x - y = -3$

We start by rewriting the second equation as $x = y - 3$. Then we substitute $y - 3$ in for x in the first equation.

$$-3(y - 3) + 3y = 4$$
$$-3y + 9 + 3y = 4$$
$$0 + 9 = 4$$
$$9 = 4$$

This equation is never true. So the original system of equations has no solution.

6. $x - y = 13$
$2x + y = 17$

Adding the 2 equations gives $3x = 30$, so $x = 10$. Substituting this information into the first equation gives the following.

$$10 - y = 13$$
$$-y = 3$$
$$y = -3$$

So the solution is $x = 10$, $y = -3$. Let's check this.

$$x - y = 13$$
$$10 - (-3) = 13$$
$$10 + 3 = 13$$
$$13 = 13$$

$$2x + y = 17$$
$$2(10) + (-3) = 17$$
$$20 - 3 = 17$$
$$17 = 17$$

7. $7x + 2y = 22$
$8x + 2y = 32$

If we subtract the first equation from the second equation, we have $1x = 10$, or just $x = 10$. Using this information in the first equation then gives the following.

$$7(10) + 2y = 22$$
$$70 + 2y = 22$$
$$2y = -48$$
$$y = -24$$

So the solution of this system is $x = 10$, $y = -24$. Let's check it.

$$7(10) + 2(-24) = 22$$
$$70 - 48 = 22$$
$$22 = 22$$

$$8(10) + 2(-24) = 32$$
$$80 - 48 = 32$$
$$32 = 32$$

8. $-4x + 4y = 18$
$-4x + 2y = 12$

Subtracting these 2 equations gives $2y = 6$, or $y = 3$. Using this information in the first equation gives the following.

$$-4x + 4(3) = 18$$
$$-4x + 12 = 18$$
$$-4x = 6$$
$$x = -6/4 = -3/2$$

Solutions

Substituting this information back into the first equation gives the following.

$$-4(-3/2) + 4y = 18$$
$$6 + 4y = 18$$
$$4y = 12$$
$$y = 3$$

So the solution of this system is $x = -3/2$, $y = 3$. Let's check.

$$-4(-3/2) + 4(3) = 18$$
$$6 + 12 = 18$$
$$18 = 18$$

$$-4(-3/2) + 2(3) = 12$$
$$6 + 6 = 12$$
$$12 = 12$$

9. $x - 4y = 3$
        $-5x + 20y = -15$

We begin by multiplying the first equation by 5. This gives us a new system of equations, below.

$$5x - 20y = 15$$
$$-5x + 20y = -15$$

We now add these 2 equations together. This gives us 0 = 0. This is true for all real numbers, so the set of solutions is the set of all real numbers.

10. $4x + 2y = -14$
        $10x - 7y = 25$

We begin by multiplying the first equation by −5/2. This gives us a new system of equations, below.

$$(-5/2)(4x + 2y) = (-5/2)(-14)$$
$$10x - 7y = 25$$

$$-10x - 5y = 35$$
$$10x - 7y = 25$$

Now we add these equations.

$$-12y = 60$$
$$y = -5$$

We plug this information into the first of the original equations to find x.

$$4x + 2(-5) = -14$$
$$4x - 10 = -14$$
$$4x = -4$$
$$x = -1$$

So our solution is $x = -1$, $y = -5$. Let's check it.

$$4x + 2y = -14$$
$$4(-1) + 2(-5) = -14$$
$$-4 - 10 = -14$$
$$-14 = -14$$

$$10(-1) - 7(-5) = 25$$
$$-10 + 35 = 25$$
$$25 = 25$$

Lesson 9

1. $x + 2y + 3z = 9$
$2x - y + z = 8$
$3x - z = 3$

Notice that the variable y is already eliminated from the third equation. So we use the first 2 equations to obtain another equation that is missing y.

$x + 2y + 3z = 9$
$2x - y + z = 8$

$x + 2y + 3z = 9$
$2(2x - y + z) = 2(8)$

$x + 2y + 3z = 9$
$4x - 2y + 2z = 16$

Now add them together to obtain

$5x + 5z = 25$.

We put this new equation with the equation $3x - z = 3$.

$5x + 5z = 25$
$3x - z = 3$

Multiply the second of these equations by 5 and add them together.

$$5x + 5z = 25$$
$$15x - 5z = 15$$
$$20x = 40$$

So $x = 2$. We place this back into $3x - z = 3$ to obtain the following.

$$3(2) - z = 3$$
$$6 - z = 3$$
$$z = 3$$

Now we have x and z. We place these values into one of the original equations to get y.

$$x + 2y + 3z = 9$$
$$2 + 2y + 3(3) = 9$$
$$2y + 11 = 9$$
$$2y = -2$$
$$y = -1$$

So the final solution is $x = 2$, $y = -1$, $z = 3$.

2. $$x - 3y = 2$$
$$x + 2y + 5z = 2$$
$$-2x + 6y + 4z = 4$$

Subtracting the first equation from the second equation gives us

$$5y + 5z = 0.$$

Adding twice the second equation to the third equation gives us

$$10y + 14z = 8.$$

So we now have the following.

$$5y + 5z = 0$$
$$10y + 14z = 8$$

Subtracting twice the first equation from the second gives us

$$4z = 8, \text{ or}$$
$$z = 2.$$

Plugging this information into $5y + 5z = 0$ gives us the following.

$$5y + 5(2) = 0$$
$$5y + 10 = 0$$
$$5y = -10$$
$$y = -2$$

Now we plug $y = -2$ and $z = 2$ into one of the original equations to find x.

$$x - 3y = 2$$
$$x - 3(-2) = 2$$
$$x + 6 = 2$$
$$x = -4$$

The final solution is $x = -4$, $y = -2$, $z = 2$.

3. $$x - y = -1$$
        $$x + y + z = 1$$
        $$-3x - z = 5$$

Adding the first 2 equations together gives us

$$2x + z = 0.$$

We can then add this equation to the third equation in the original system to get

$$-x = 5, \text{ or}$$
$$x = -5.$$

Plugging this information into the first equation in the original system gives us the following.

$$-5 - y = -1$$
$$-y = 4$$
$$y = -4$$

Lastly, plugging $x = -5$ into the third equation in the original system gives us the below.

$$-3(-5) - z = 5$$
$$15 - z = 5$$
$$-z = -10$$
$$z = 10$$

The solution of the system is $x = -5$, $y = -4$, $z = 10$.

4. $x - y + z = 7$
      $3x + y + 6z = 1$
      $-2x + 2y - 2z = 5$

Adding the first 2 equations gives us $4x + 7y = 8$.

Adding 2 times the first equation to the third equation gives us

$$0 + 0 + 0 = 19, \text{ or}$$
$$0 = 19.$$

This is never true, so the original system has no solution (it is inconsistent).

5. $x + y + z = 6$
      $x - y - z = 0$
      $2x - 3y + 5z = 5$

Adding the first 2 equations gives us

$$2x = 6, \text{ or}$$
$$x = 3.$$

Adding 5 times the second equation to the third equation gives us

$$7x - 8y = 5.$$

Since $x = 3$, we can plug that in.

$$7(3) - 8y = 5$$
$$21 - 8y = 5$$
$$-8y = -16$$
$$y = 2$$

Plugging $x = 3$ and $y = 2$ into the first equation in the original system gives us the below.

$$3 + 2 + z = 6$$
$$5 + z = 6$$
$$z = 1$$

So the solution of the system is $x = 3$, $y = 2$, $z = 1$.

1. $y < x - 2$
           $y \geq 2x - 3$

The 2 boundary lines are drawn as follows.
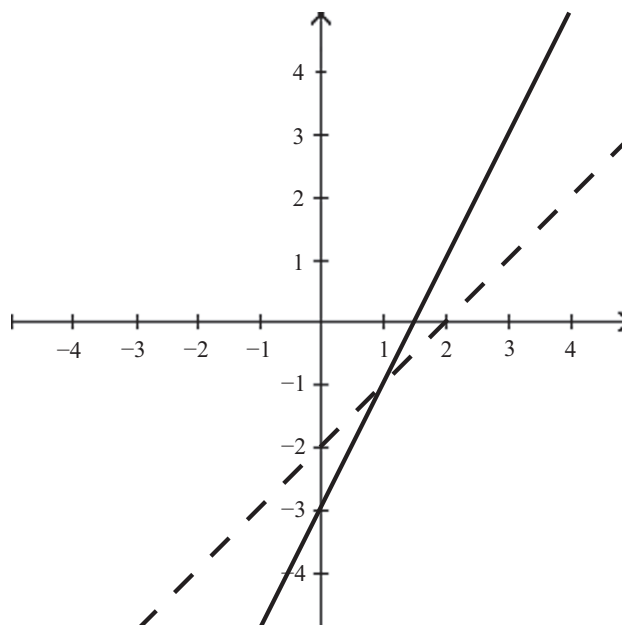
We then shade the solution sets of the 2 inequalities, and the final answer is the intersection.
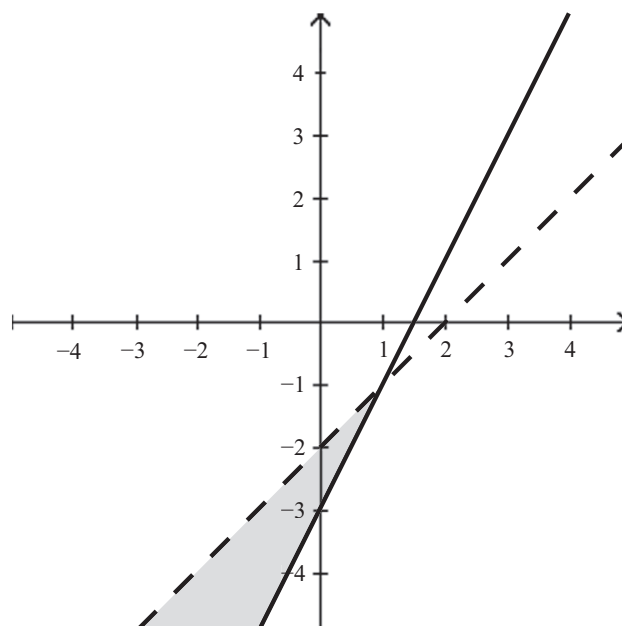
2. $y > 4x$
        $y > 2$

The 2 boundary lines are drawn as follows.
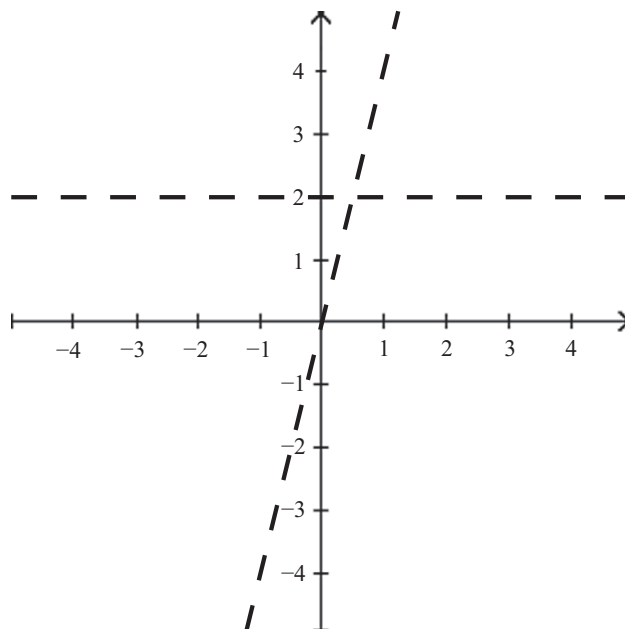
We then shade the solution sets of the 2 inequalities, and the final answer is the intersection.
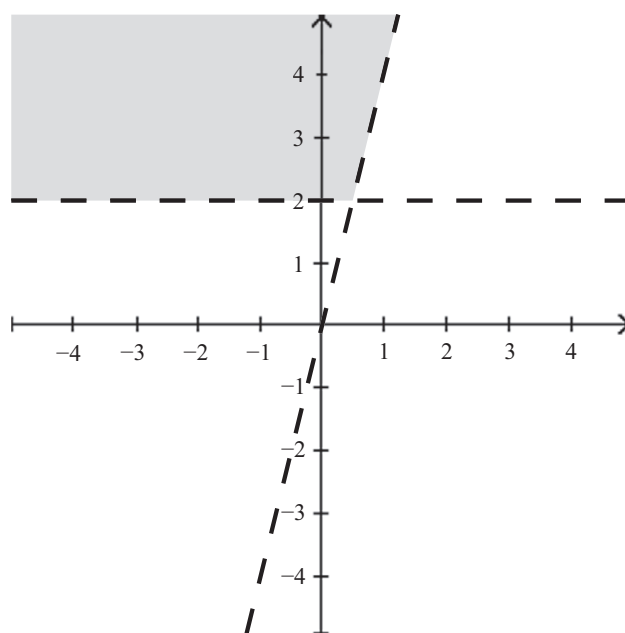
3. $y \le -2x + 1$
      $y > x - 5$

The 2 boundary lines are drawn as follows.
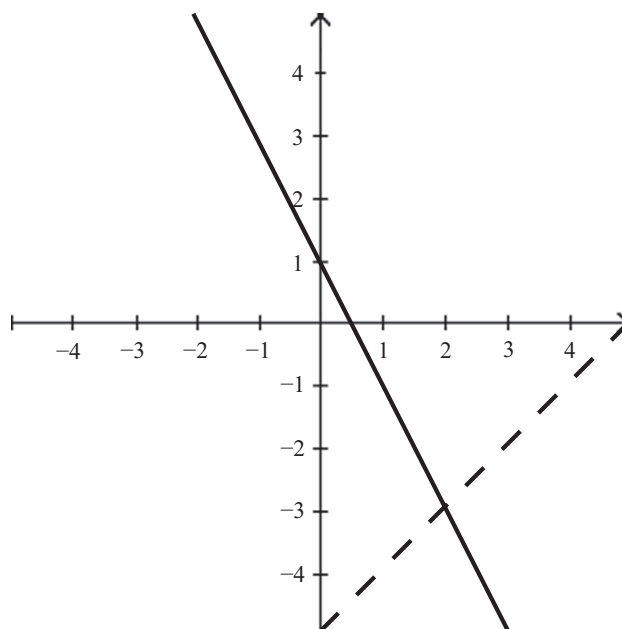
We then shade the solution sets of the 2 inequalities, and the final answer is the intersection.
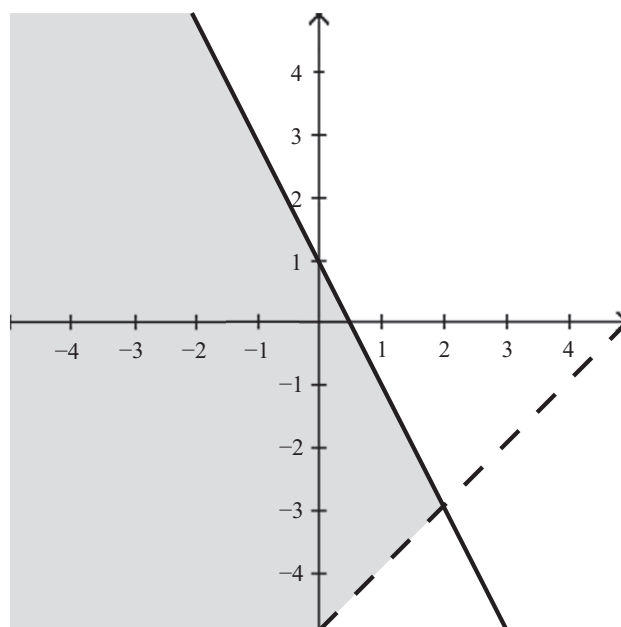
4. $y \le -3x + 7$
           $y \ge 2x - 6$

The 2 boundary lines are drawn as follows.
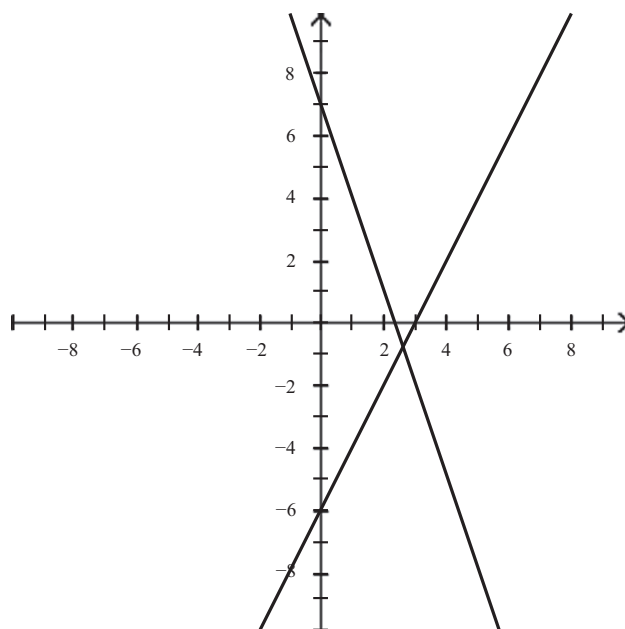
We then shade the solution sets of the 2 inequalities, and the final answer is the intersection.
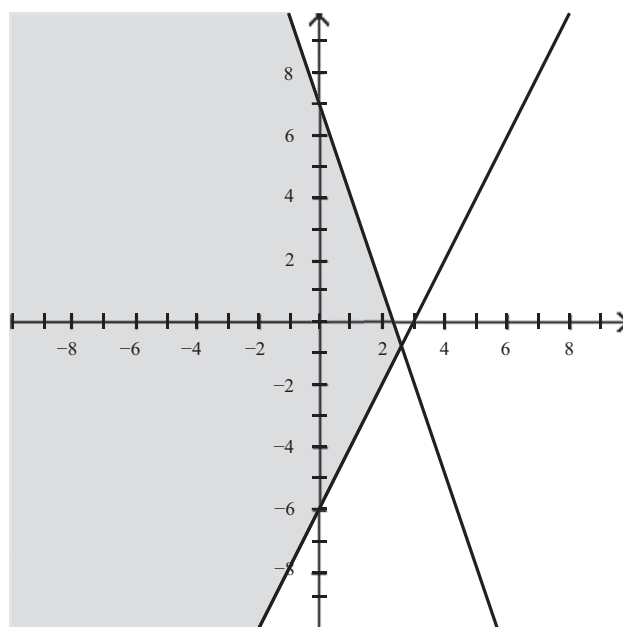
5. $y \leq 4$
          $y \leq -3x + 2$

The 2 boundary lines are drawn as follows.
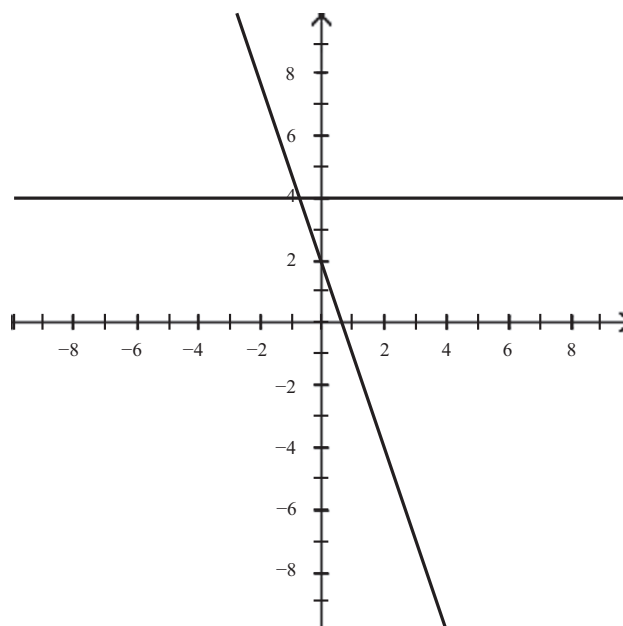
We then shade the solution sets of the 2 inequalities, and the final answer is the intersection.
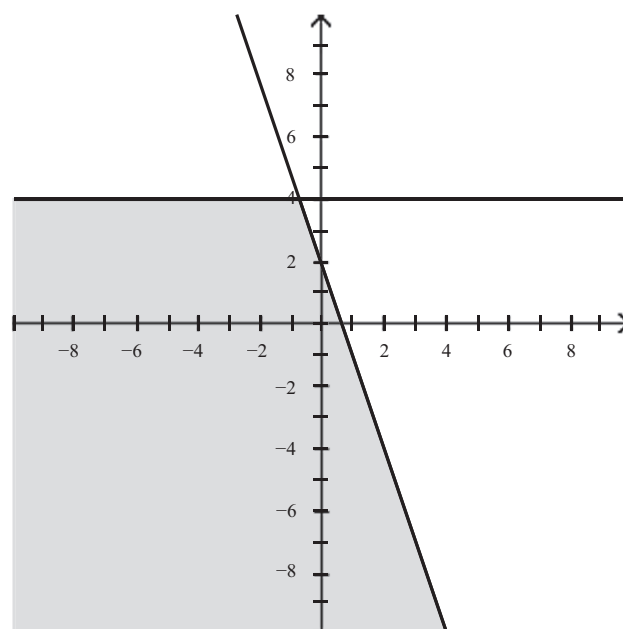
6. $\quad y < -2x - 4$
$\qquad y < 6x + 2$

The 2 boundary lines are drawn as follows.
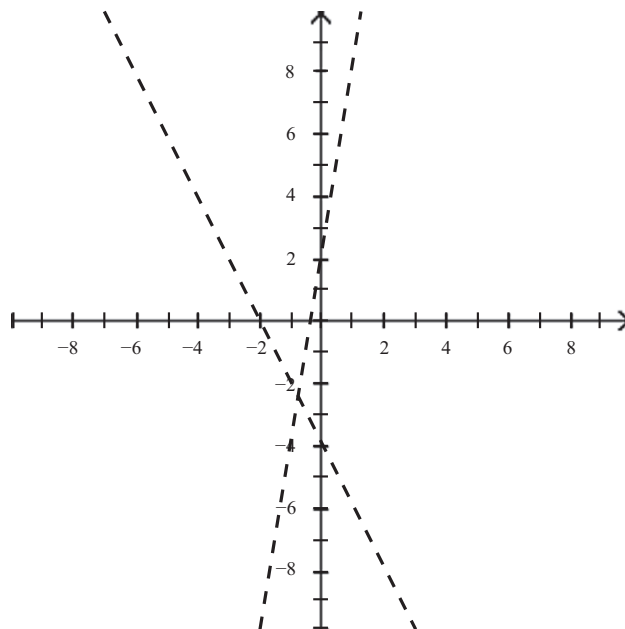
We then shade the solution sets of the 2 inequalities, and the final answer is the intersection.
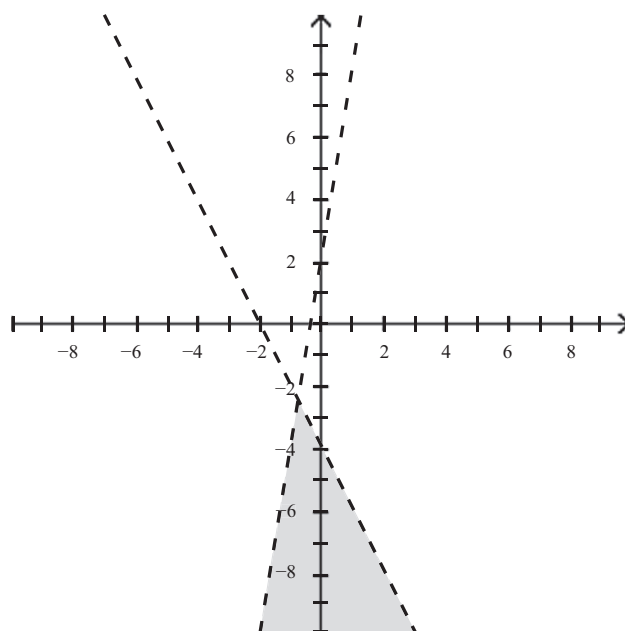
7. $y \geq -x$
           $y \geq 2x$

The 2 boundary lines are drawn as follows.
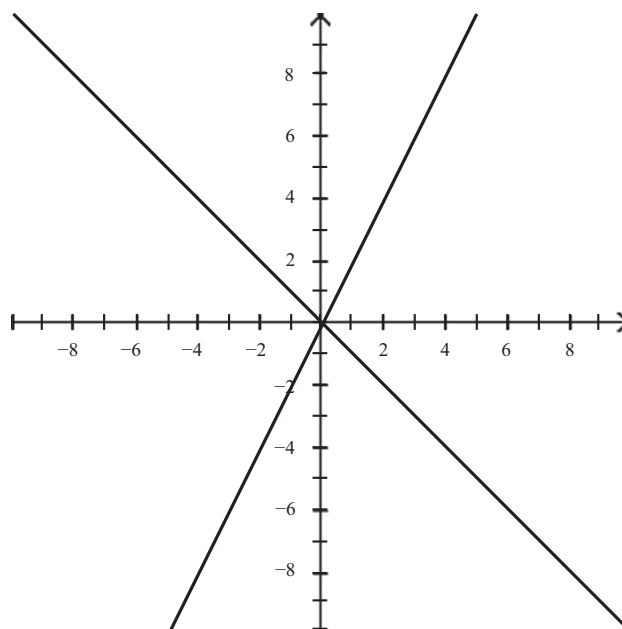
We then shade the solution sets of the 2 inequalities, and the final answer is the intersection.
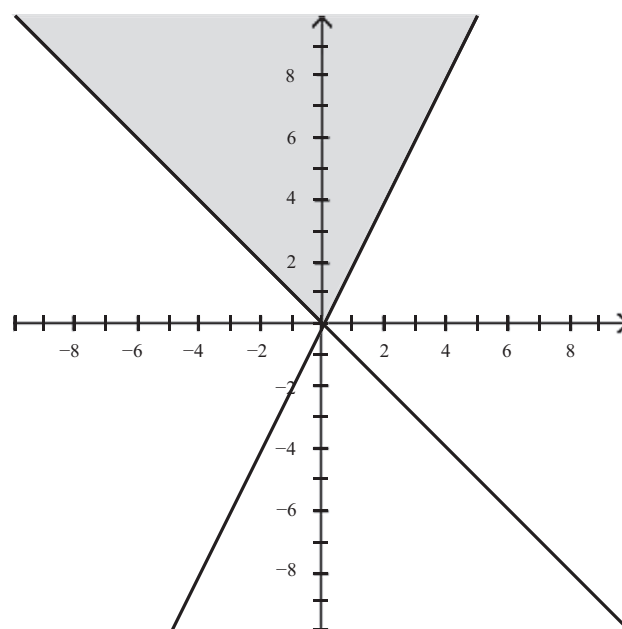
8. $y > -2$
        $y \geq 3$

The 2 boundary lines are drawn as follows.
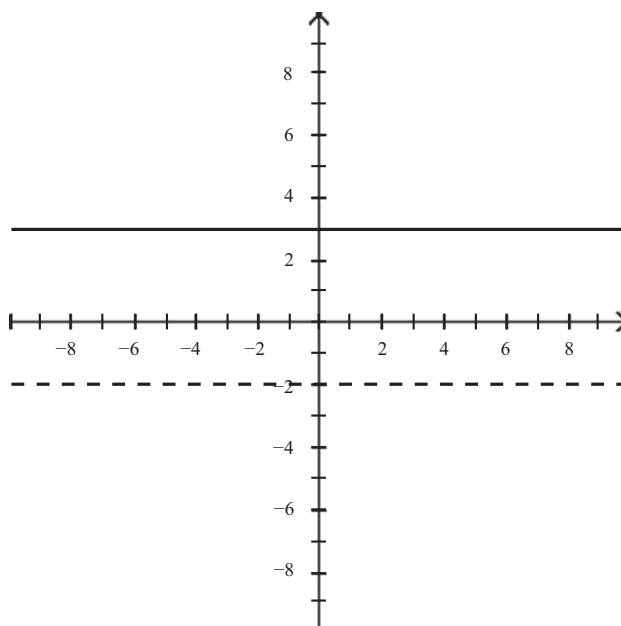
We then shade the solution sets of the 2 inequalities, and the final answer is the intersection.
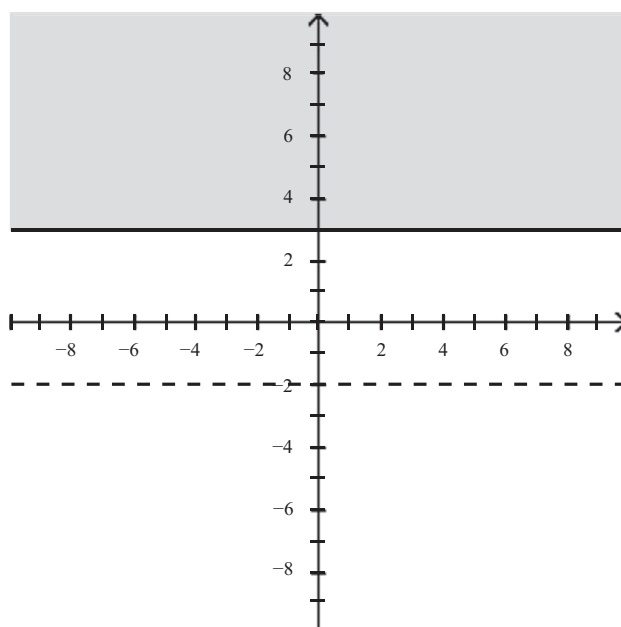
9. $y \leq 4x - 3$
       $y \geq -3x + 7$

The 2 boundary lines are drawn as follows.
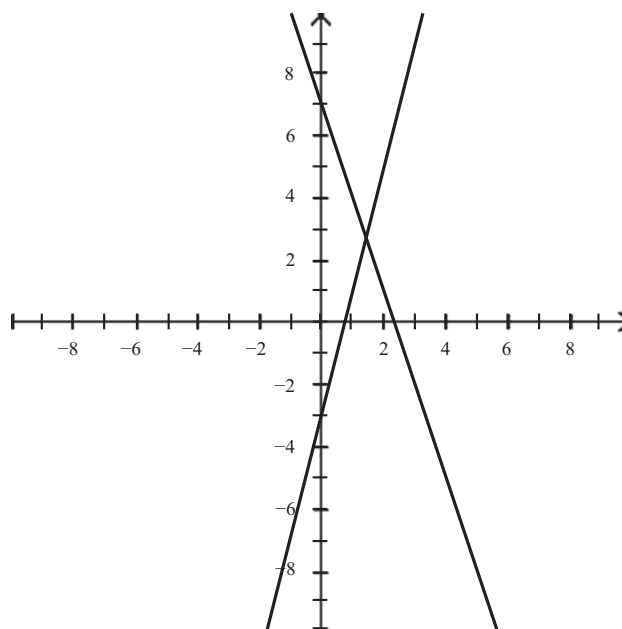
We then shade the solution sets of the 2 inequalities, and the final answer is the intersection.
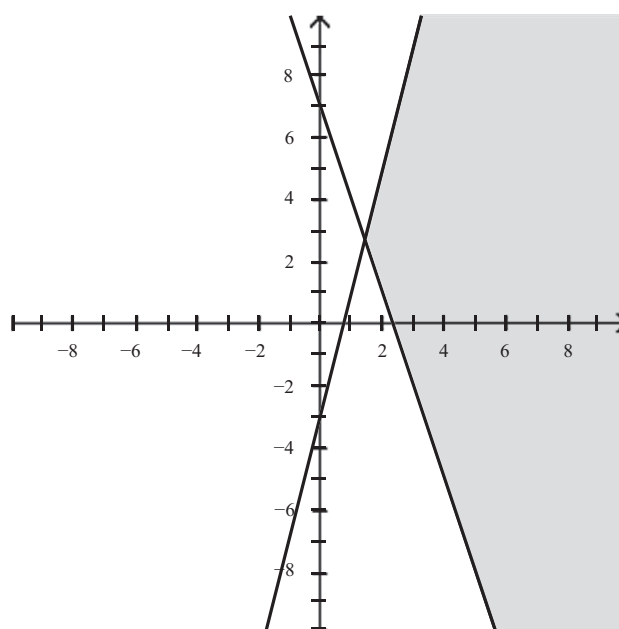
10. $y > 5x$
         $y < 2x$

The 2 boundary lines are drawn as follows.
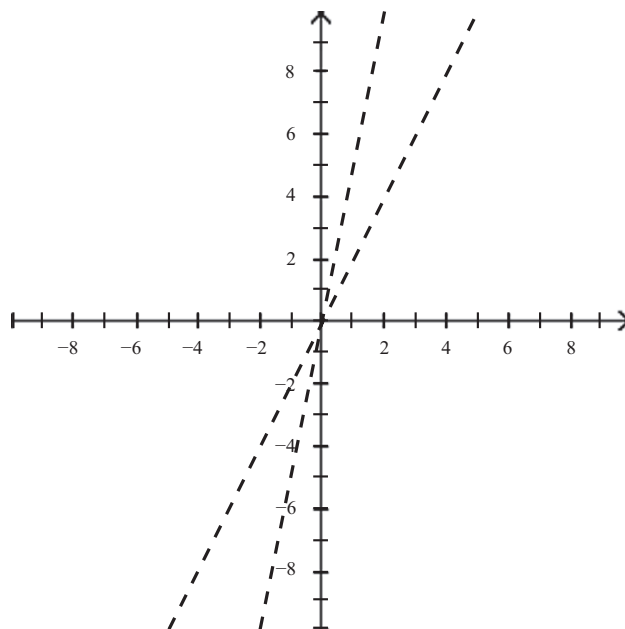
We then shade the solution sets of the 2 inequalities, and the final answer is the intersection.
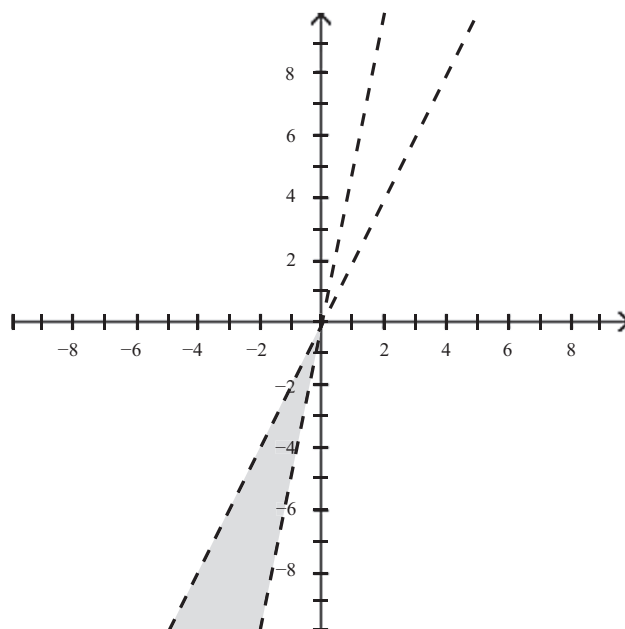
1. $f(x) = x^2 - 5$

We begin with the graph of x^2 and shift it 5 units down.
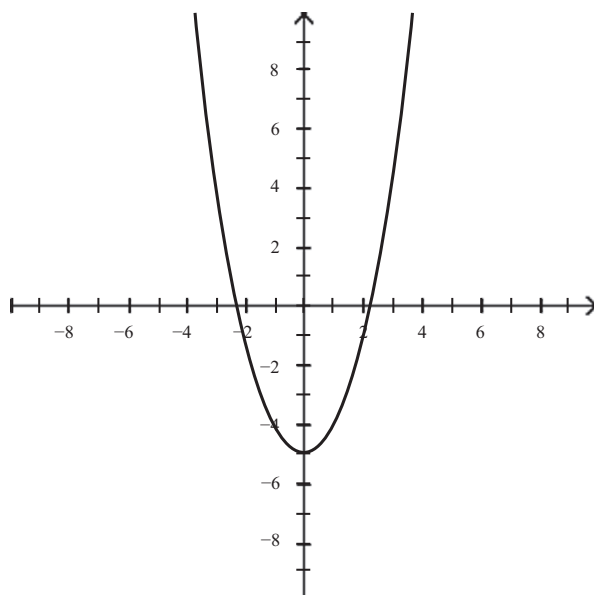
2. $f(x) = (x - 4)^2$

We begin with the graph of x^2 and shift it 4 units to the right.
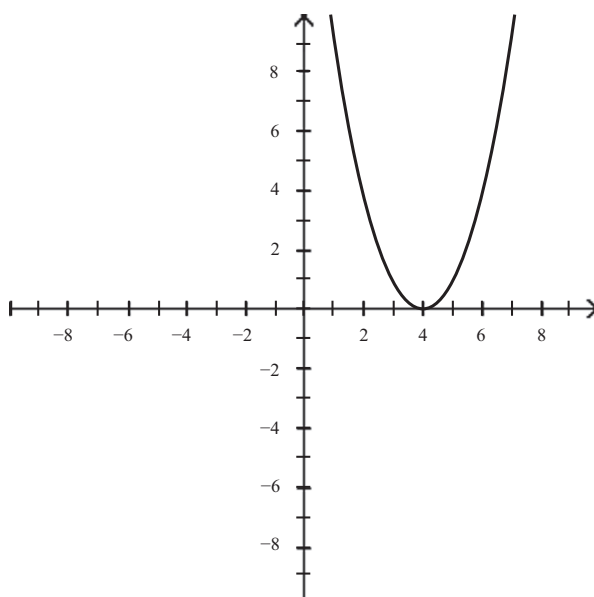
3. $f(x) = (x+1)^2$

We begin with the graph of x^2 and shift it 1 unit to the left.
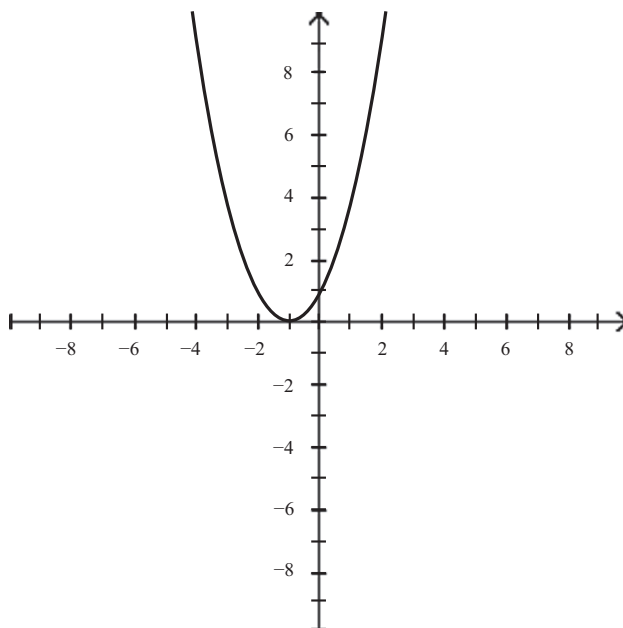
4. $f(x) = x^2 + 2$

We begin with the graph of x^2 and shift it 2 units up.
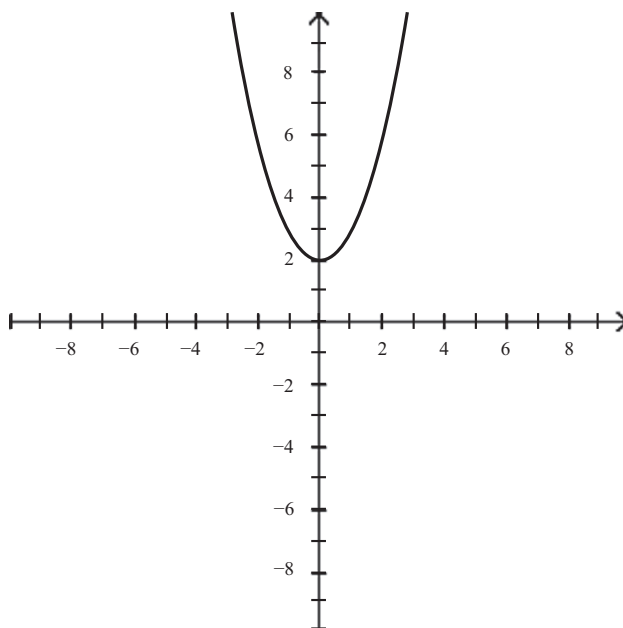
5. $f(x) = (x-3)^2 - 1$

We begin with the graph of x^2 and shift it 3 units to the right, then 1 unit down.
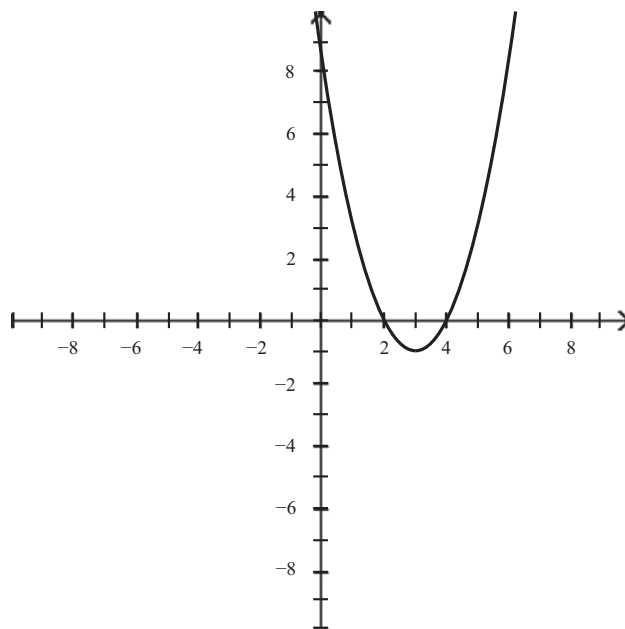
6. $f(x) = (x+2)^2 + 3$

We begin with the graph of x^2 and shift it 2 units to the left, then 3 units up.
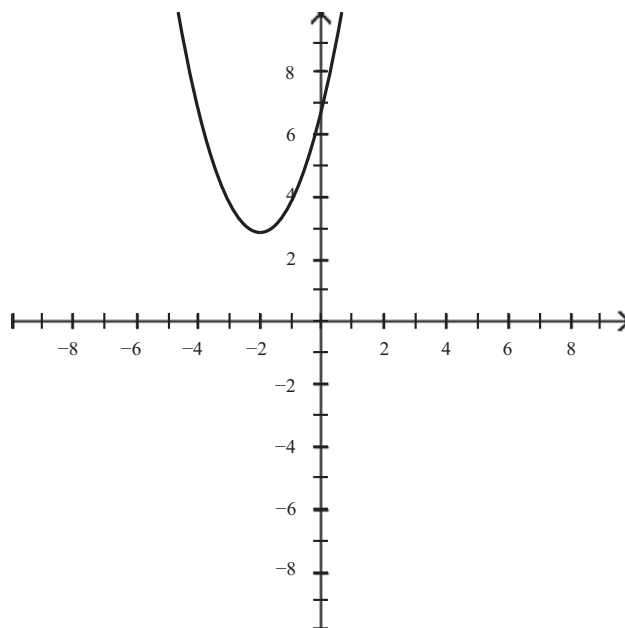
7. $f(x) = (x+4)^2 - 5$

We begin with the graph of x^2 and shift it 4 units to the left, then 5 units down.
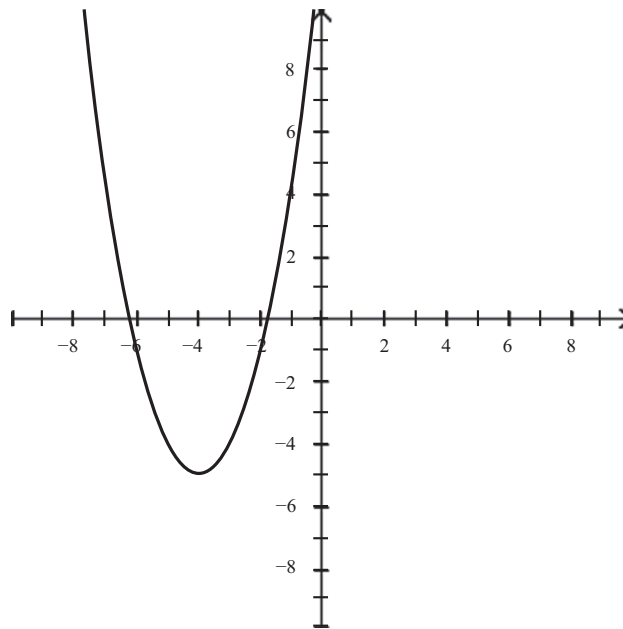
8. $f(x) = -x^2$

We begin with the graph of x^2 and flip it around the *x*-axis.
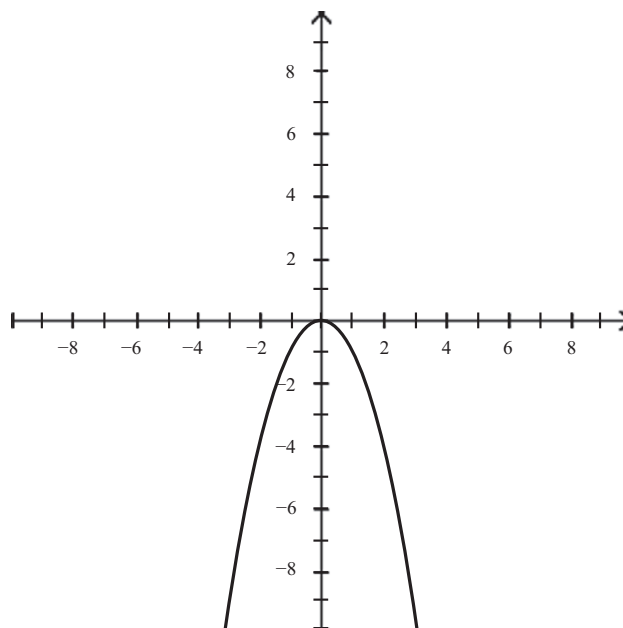
9. $f(x) = -x^2 + 2$

We begin with the graph of x^2, flip it around the x-axis, and then shift it 2 units up.
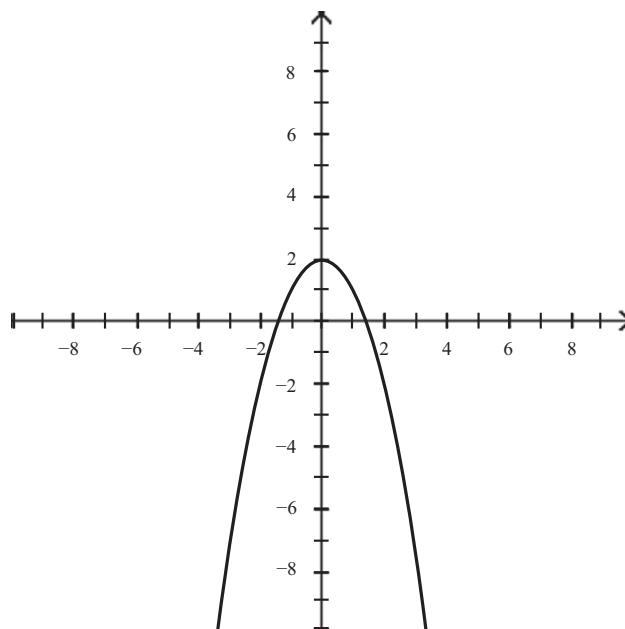
10. $f(x) = -(x - 3)^2$

We begin with the graph of x^2, shift it 3 units to the right, and then flip it around the x-axis.
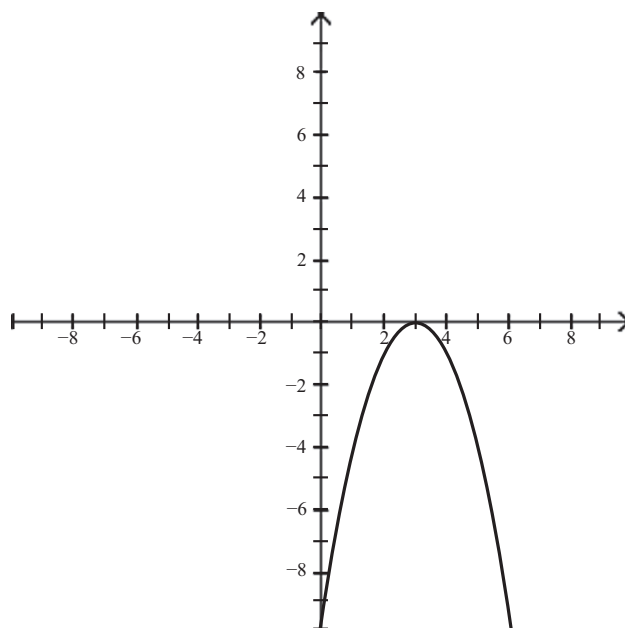
1. $x^2 + 8x + 15$
   $(x+5)(x+3)$

2. $y^2 + 2y - 24$
   $(y+6)(y-4)$

3. $x^2 - x - 42$
   $(x-7)(x+6)$

4. $x^2 - 14x + 40$
   $(x-10)(x-4)$

5. $5t^2 - 30t + 40$
   $5(t^2 - 6t + 8)$
   $5(t-4)(t-2)$

6. $x^2 + 18 = 11x - 6$

First, move all the terms to the left-hand side of the equation.

$$x^2 - 11x + 24 = 0$$

Now factor the left-hand side of the equation.

$$(x-8)(x-3) = 0$$

This means $x - 8 = 0$ or $x - 3 = 0$. So $x = 8$ and $x = 3$ are the solutions.

7. $x^2 + 24 = 10x$

First, move all the terms to the left-hand side of the equation.

$$x^2 - 10x + 24 = 0$$

Now factor the left-hand side of the equation.

$$(x-6)(x-4) = 0$$

This means $x - 6 = 0$ or $x - 4 = 0$. So $x = 6$ and $x = 4$ are the solutions.

8. $x^2 = 18 - 3x$

First, move all the terms to the left-hand side of the equation.

$x^2 + 3x - 18 = 0$

Now factor the left-hand side of the equation.

$(x+6)(x-3) = 0$

This means $x + 6 = 0$ or $x - 3 = 0$. So $x = -6$ and $x = 3$ are the solutions.

9. $3x^2 - 12x - 7 = 4x + 5$

First, move all the terms to the left-hand side of the equation.

$3x^2 - 16x - 12 = 0$

Now factor the left-hand side of the equation.

$(3x+2)(x-6) = 0$

This means $3x + 2 = 0$ or $x - 6 = 0$. So $x = -\dfrac{2}{3}$ and $x = 6$ are the solutions.

10. $x^2 - 10x = -25$

First, move all the terms to the left-hand side of the equation.

$x^2 - 10x + 25 = 0$

Now factor the left-hand side of the equation.

$(x-5)(x-5) = 0$

This means $x - 5 = 0$. So $x = 5$ is the solution (and the only solution).

1. $x^2 - 169 = 0$
      $x^2 = 169$
      $x = \pm\sqrt{169}$
      $x = \pm 13$

The solutions are $x = 13$ and $x = -13$.

2. $x^2 + 36 = 0$
      $x^2 = -36$
      $x = \pm\sqrt{-36}$
      $x = \pm 6i$

The solutions are $x = 6i$ and $x = -6i$. (Notice that these are not real number solutions, so if the problem had asked for real number solutions, then the final answer would have been that there are no solutions.)

3. $x^2 - 3 = 61$
      $x^2 = 64$
      $x = \pm 8$

The solutions are $x = 8$ and $x = -8$.

4. $4x^2 = 25$
      $\sqrt{4x^2} = \pm\sqrt{25}$
      $2x = \pm 5$
      $x = \pm\dfrac{5}{2}$

So the solutions are $x = \dfrac{5}{2}$ and $x = -\dfrac{5}{2}$.

5.
$$7x^2 - 1 = 27$$
$$7x^2 = 28$$
$$x^2 = 4$$
$$x = \pm 2$$

So the solutions are $x = 2$ and $x = -2$.

6.
$$8x^2 - 23 = 177$$
$$8x^2 = 200$$
$$x^2 = 25$$
$$x = \pm 5$$

So the solutions are $x = 5$ and $x = -5$.

7.
$$(x + 7)^2 - 9 = 0$$
$$(x + 7)^2 = 9$$
$$x + 7 = \pm 3$$

So we know that $x + 7 = 3$ or $x + 7 = -3$. That means the solutions are $x = -4$ and $x = -10$.

8.
$$(2x - 3)^2 - 225 = 0$$
$$(2x - 3)^2 = 225$$
$$2x - 3 = \pm\sqrt{225}$$
$$2x - 3 = \pm 15$$

So $2x - 3 = 15$ or $2x - 3 = -15$. This means $2x = 18$ or $2x = -12$. This implies that the solutions are $x = 9$ and $x = -6$.

9.
$$9(x + 10)^2 + 121 = 0$$
$$9(x + 10)^2 = -121$$
$$\sqrt{9(x + 10)^2} = \pm\sqrt{-121}$$
$$3(x + 10) = \pm 11i$$

Notice that we will not have any real number solutions (thanks to the presence of that i). We know that $3(x + 10) = 11i$ or $3(x + 10) = -11i$. This means that $x + 10 = \frac{11}{3}i$ or $x + 10 = -\frac{11}{3}i$. So our solutions are $x = -10 + \frac{11}{3}i$ and $x = -10 - \frac{11}{3}i$.

10.
$$4(x-2)^2 - 289 = 0$$
$$4(x-2)^2 = 289$$
$$\sqrt{4(x-2)^2} = \pm\sqrt{289}$$
$$2(x-2) = \pm 17$$
$$x - 2 = \pm\frac{17}{2}$$
$$x = 2 \pm \frac{17}{2}$$

So our solutions are $x = 2 + \frac{17}{2}$ and $x = 2 - \frac{17}{2}$.

Lesson 14

1. $f(x) = x^2 - 20x + 103$

$$x^2 - 20x + 103$$
$$= x^2 - 20x + 100 - 100 + 103$$
$$= (x-10)^2 - 100 + 103$$
$$= (x-10)^2 + 3$$

So the vertex is at (10, 3).

2. $f(x) = x^2 + 14x + 32$

$$x^2 + 14x + 32$$
$$= x^2 + 14x + 49 - 49 + 32$$
$$= (x+7)^2 - 49 + 32$$
$$= (x+7)^2 - 17$$

So the vertex is at (−7, −17).

3. $f(x) = (x+3)^{1/4} + 2$

$$x^2 + 2x - 7$$
$$= x^2 + 2x + 1 - 1 - 7$$
$$= (x+1)^2 - 1 - 7$$
$$= (x+1)^2 - 8$$

So the vertex is at (−1, −8).

4. $f(x) = x^2 - 16x + 64$

$$x^2 - 16x + 64$$
$$= x^2 - 16x + 64 - 64 + 64$$
$$= (x-8)^2 - 64 + 64$$
$$= (x-8)^2 + 0$$

So the vertex is at (8, 0).

5. $f(x) = -(x-2)^{1/3} + 4$

This equation is already in the correct form, if we realize that it is the same as $f(x) = (x+3)^{1/4} + 2$. Therefore, the vertex is located at (0, 3).

6. $x^2 - 6x = -5$

Completing the square gives us the following.

$$x^2 - 6x = -5$$
$$x^2 - 6x + 9 - 9 = -5$$
$$(x-3)^2 - 9 = -5$$
$$(x-3)^2 = 4$$

Now we take square roots and interpret our results.

$$(x-3)^2 = 4$$
$$x - 3 = \pm 2$$

So $x - 3 = 2$ or $x - 3 = -2$. This means our solutions are $x = 5$ and $x = 1$.

7. $x^2 - 7x = 3x - 9$

We first move the $3x$ term to the left-hand side and simplify.

$$x^2 - 10x = -9$$

Completing the square gives us the following.

$$x^2 - 10x + 25 - 25 = -9$$
$$(x-5)^2 - 25 = -9$$
$$(x-5)^2 = 25 - 9$$
$$(x-5)^2 = 16$$

Now we take square roots and interpret our results.

$$(x-3)^2 = 4$$
$$x - 3 = \pm 2$$

This is the same as $x - 5 = 4$ or $x - 5 = -4$, which gives us the solutions $x = 9$ and $x = 1$.

8. $x^2 - 12x = -18x - 16$

We first move the $-18x$ term to the left-hand side and simplify.

$$x^2 + 6x = -16$$

Completing the square gives us the following.

$$x^2 + 6x = -16$$
$$x^2 + 6x + 9 - 9 = -16$$
$$(x+3)^2 - 9 = -16$$
$$(x+3)^2 = -7$$

Note that this implies that there are no real number solutions, because the square of any real number can never equal -7. However, if we are willing to have complex number solutions, then we can proceed as in the other examples.

$$(x+3)^2 = -7$$
$$x + 3 = \pm\sqrt{-7}$$
$$x + 3 = \pm\sqrt{7}i$$

This means that we have $x + 3 = \sqrt{7}i$ or $x + 3 = -\sqrt{7}i$, which implies that our complex number solutions are $x = -3 + \sqrt{7}i$ and $x = -3 - \sqrt{7}i$.

9. $3x^2 - 6x = 2x^2 - 7$

Subtracting $2x^2$ from both sides of the equation gives

$x^2 - 6x = -7$.

Completing the square gives us the following.

$$x^2 - 6x + 9 - 9 = -7$$
$$(x-3)^2 - 9 = -7$$
$$(x-3)^2 = 9 - 7$$
$$(x-3)^2 = 2$$

Now we take square roots and interpret our results.

$$x - 3 = \pm\sqrt{2}$$

So $x - 3 = \sqrt{2}$ or $x - 3 = -\sqrt{2}$, which means our solutions are $x = 3 + \sqrt{2}$ and $x = 3 - \sqrt{2}$.

10. $-x^2 + 16x = -36$

We begin by multiplying the entire equation by −1.

$x^2 - 16x = 36$

The solutions of this equation are the same as the solutions of the original equation. Completing the square gives us the following.

$$x^2 - 16x + 64 - 64 = 36$$
$$(x-8)^2 - 64 = 36$$
$$(x-8)^2 = 100$$

Now we take square roots and interpret our results.

$$x - 8 = \pm\sqrt{100}$$
$$x - 8 = \pm 10$$

Therefore, $x - 8 = 10$ or $x - 8 = -10$. This means our solutions are $x = 18$ and $x = -2$.

1. $x^2 + 9x + 20 = 0$

Using the quadratic formula (with a = 1, b = 9, and c = 20), we know that $x = \dfrac{-9 + \sqrt{(9)^2 - 4(1)(20)}}{2(1)}$ and $x = \dfrac{-9 - \sqrt{(9)^2 - 4(1)(20)}}{2(1)}$. This is the same as $x = \dfrac{-9 + \sqrt{1}}{2}$ and $x = \dfrac{-9 - \sqrt{1}}{2}$. We can simplify these to $x = -4$ and $x = -5$. These are the 2 solutions.

2. $x^2 - 8x + 16 = 0$

Using the quadratic formula (with a = 1, b = −8, and c = 16), we know that $x = \dfrac{-(-8) + \sqrt{(-8)^2 - 4(1)(16)}}{2(1)}$ and $x = \dfrac{-(-8) - \sqrt{(-8)^2 - 4(1)(16)}}{2(1)}$. This is the same as $x = \dfrac{8 + \sqrt{0}}{2}$ and $x = \dfrac{8 - \sqrt{0}}{2}$. We can simplify these to $x = 4$ and $x = 4$, which is the same as just $x = 4$. So there is really only one solution.

3. $5x^2 + 11x + 2 = 0$

Using the quadratic formula (with a = 5, b = 11, and c = 2), we know that $x = \dfrac{-11 + \sqrt{(11)^2 - 4(5)(2)}}{2(5)}$ and $x = \dfrac{-11 - \sqrt{(11)^2 - 4(5)(2)}}{2(5)}$. This is the same as $x = \dfrac{-11 + \sqrt{81}}{10}$ and $x = \dfrac{-11 - \sqrt{81}}{10}$. We can simplify these to $x = -\dfrac{2}{10}$ and $x = -\dfrac{20}{10}$, which is the same as $x = -\dfrac{1}{5}$ and $x = -2$. These are the 2 solutions.

4. $6x^2 - 17x + 9 = 4x^2 - 14x + 3$

We begin by noting that the original equation is equivalent to $2x^2 - 3x + 6 = 0$. Using the quadratic formula (with $a = 2$, $b = -3$, and $c = 6$), we know that $x = \dfrac{-(-3) + \sqrt{(-3)^2 - 4(2)(6)}}{2(2)}$ and $x = \dfrac{-(-3) - \sqrt{(-3)^2 - 4(2)(6)}}{2(2)}$. This is the same as $x = \dfrac{3 + \sqrt{-39}}{4}$ and $x = \dfrac{3 - \sqrt{-39}}{4}$. Because the numbers under the square root symbol are negative, we know that there are no real number solutions to the original equation.

5. $-x^2 - 4x + 7 = 0$

Using the quadratic formula (with $a = -1$, $b = -4$, and $c = 7$), we know that $x = \dfrac{-(-4) + \sqrt{(-4)^2 - 4(-1)(7)}}{2(-1)}$ and $x = \dfrac{-(-4) - \sqrt{(-4)^2 - 4(-1)(7)}}{2(-1)}$. This is the same as $x = \dfrac{4 + \sqrt{44}}{-2}$ and $x = \dfrac{4 - \sqrt{44}}{-2}$. We can simplify these to $x = \dfrac{4 + 2\sqrt{11}}{-2}$ and $x = \dfrac{4 - 2\sqrt{11}}{-2}$, which is the same as $x = -2 - \sqrt{11}$ and $x = -2 + \sqrt{11}$. These are the 2 solutions.

6. $x^2 + 6x + 9 = 0$

The discriminant of this quadratic equation is the following.

$$6^2 - 4(1)(9) = 36 - 36 = 0$$

Because the discriminant equals 0, we know that the equation has exactly one real number solution.

7. $x^2 + 7x - 13 = 0$

The discriminant of this quadratic equation is the following.

$$7^2 - 4(1)(-13) = 49 + 52 = 101$$

Because the discriminant is positive, we know that the equation has exactly 2 real number solutions.

8. $x^2 + 9x + 21 = 0$

The discriminant of this quadratic equation is the following.

$$9^2 - 4(1)(21) = 81 - 84 = -3$$

Because the discriminant is negative, we know that the equation has no real number solutions.

9. $3x^2 + 9x - 5 = 0$

The discriminant of this quadratic equation is the following.

$$9^2 - 4(3)(-5) = 81 + 60 = 141$$

Because the discriminant is positive, we know that the equation has exactly 2 real number solutions.

10. $-x^2 + 3x - 5 = 0$

The discriminant of this quadratic equation is the following.

$$3^2 - 4(-1)(-5) = 9 - 20 = -11$$

Because the discriminant is negative, we know that the equation has no real number solutions.

1. $y \geq -x^2 + 1$

We first draw the boundary graph.
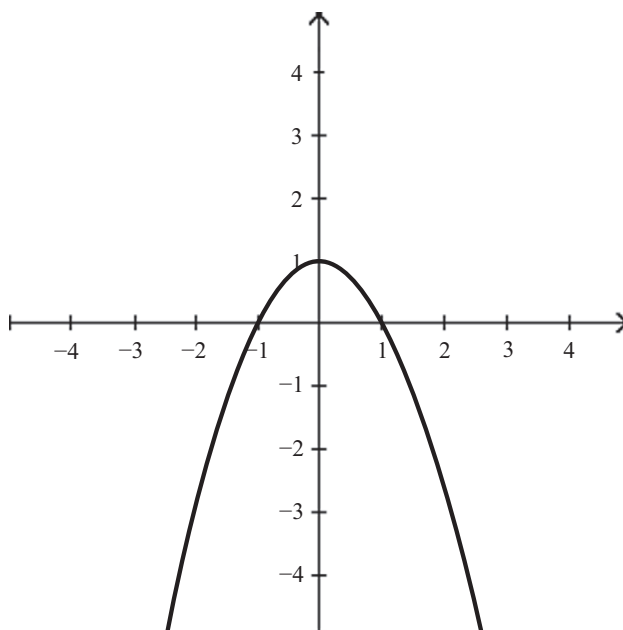
Then we shade in the appropriate portion of the xy-plane that shows the solution set for the inequality.
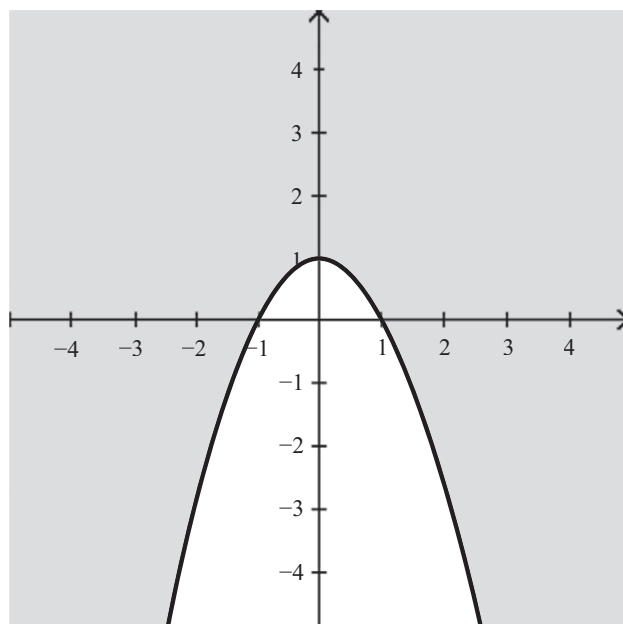
2. $y > 5x^2$

We first draw the boundary graph.
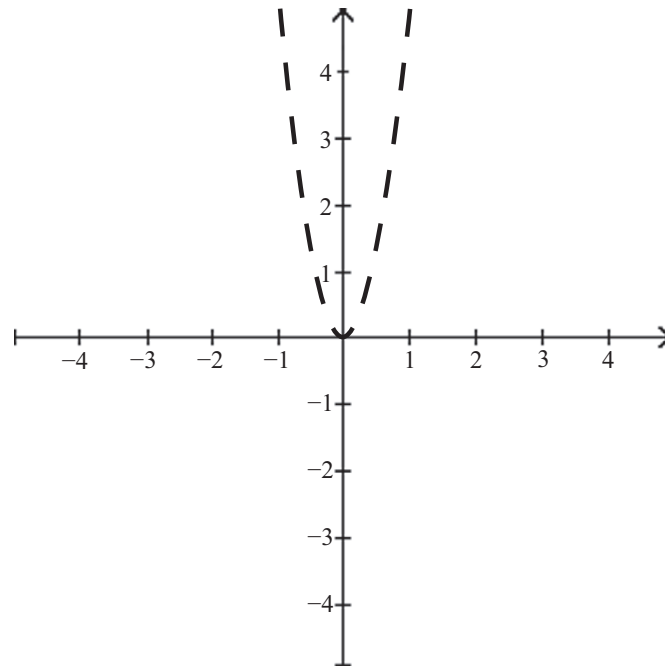
Then we shade in the appropriate portion of the *xy*-plane that shows the solution set for the inequality.
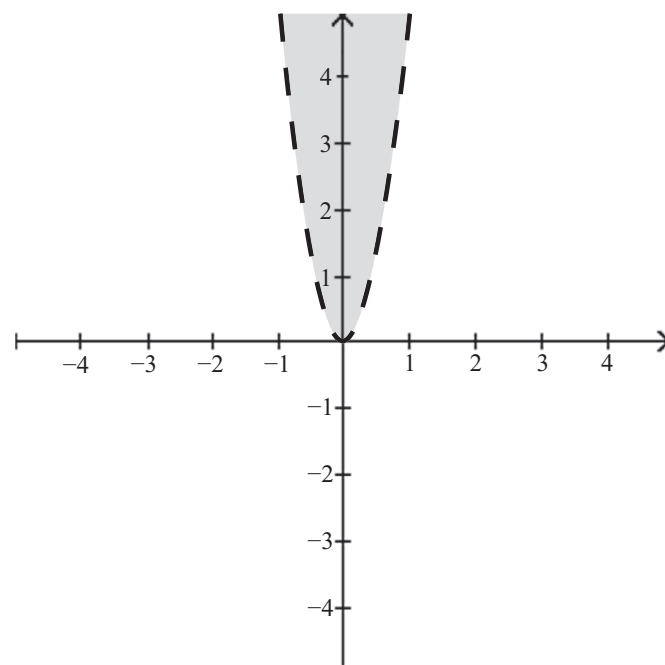
3. $y \leq x^2 - 11x + 30$

We first draw the boundary graph.
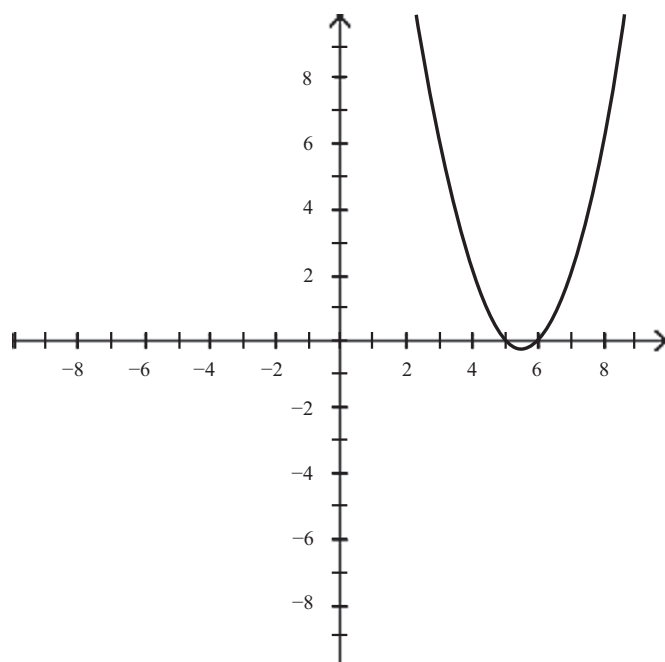
Then we shade in the appropriate portion of the xy-plane that shows the solution set for the inequality.
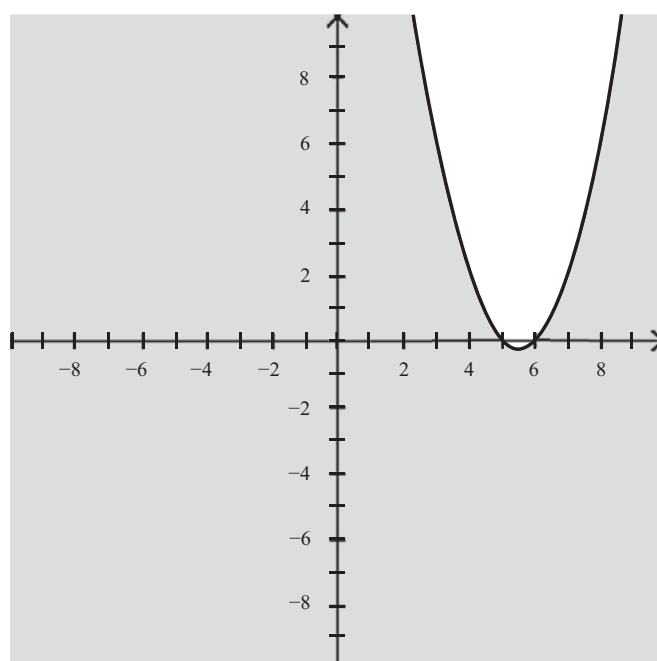
4. $y < x^2 - 4x + 6$

We first draw the boundary graph.
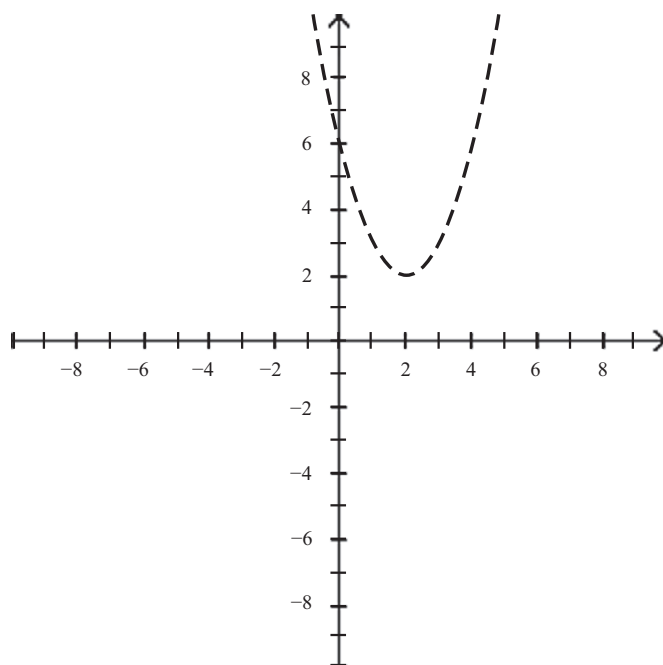
Then we shade in the appropriate portion of the xy-plane that shows the solution set for the inequality.
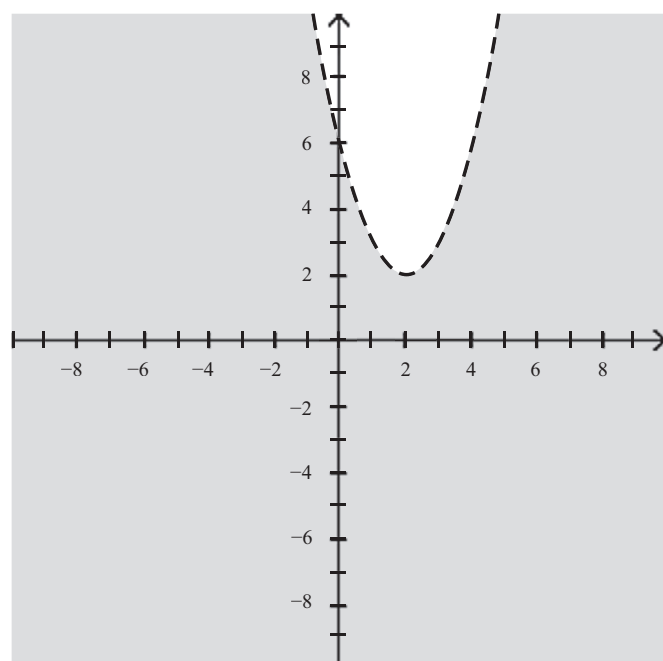
5. $y > x^2 - 13x + 40$

We first draw the boundary graph.
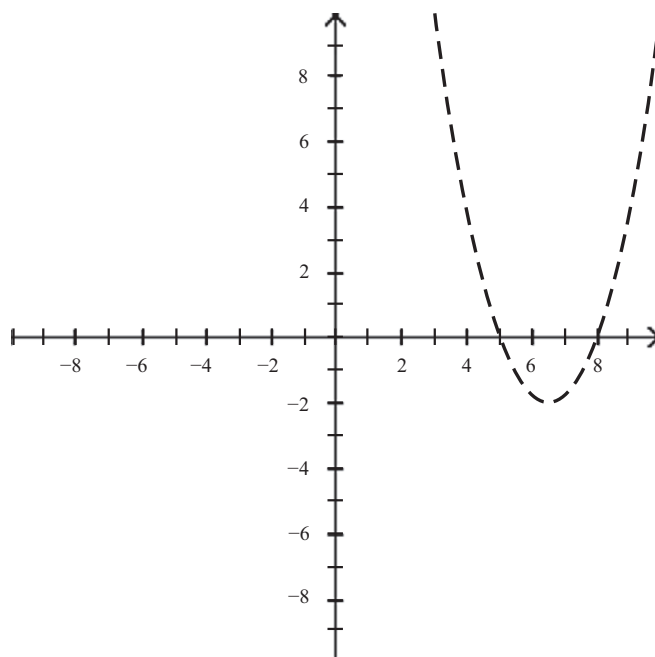
Then we shade in the appropriate portion of the xy-plane that shows the solution set for the inequality.
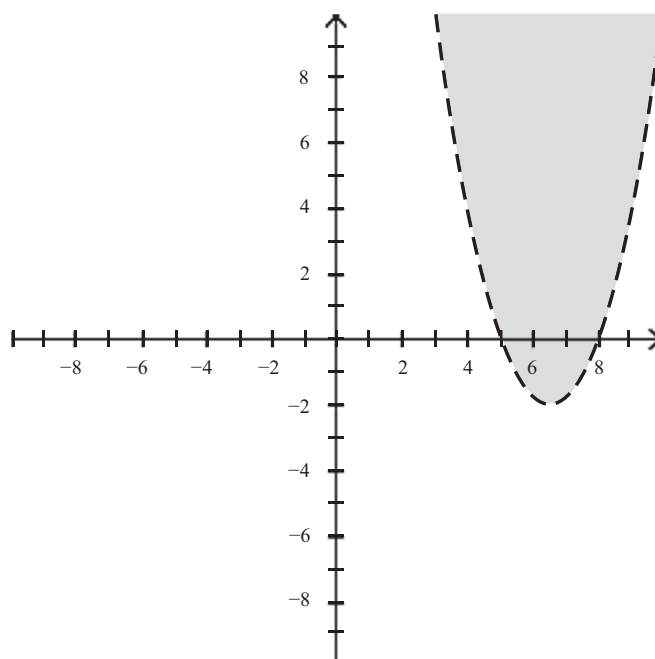
6. $y \ge x^2 - 1$

$y < 1 - x^2$

We sketch the graph of the parabola that serves as the boundary of the first inequality.
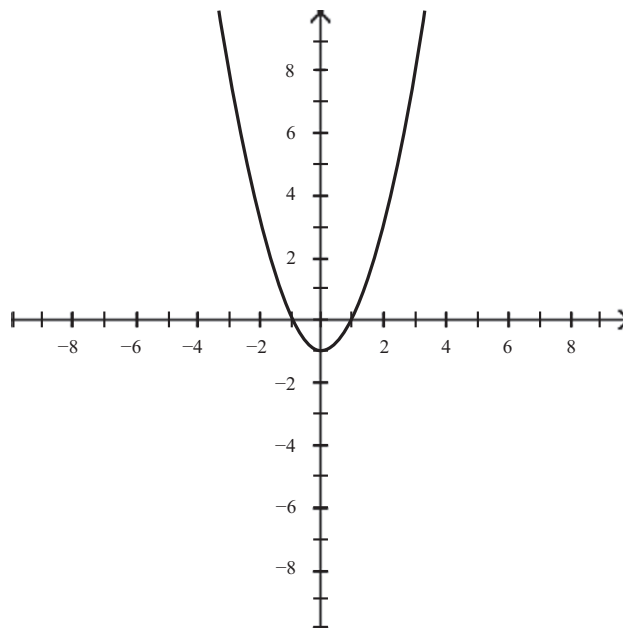
We then shade in the solution set of the first inequality.
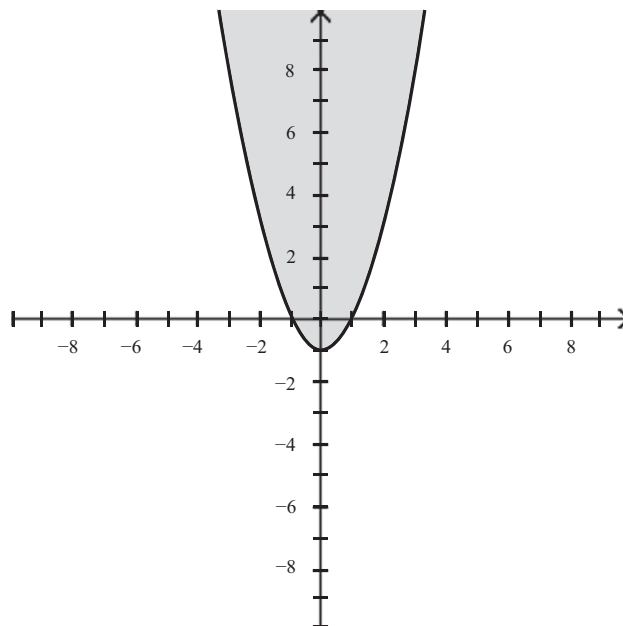
Next, we sketch the graph of the parabola that serves as the boundary of the second inequality.
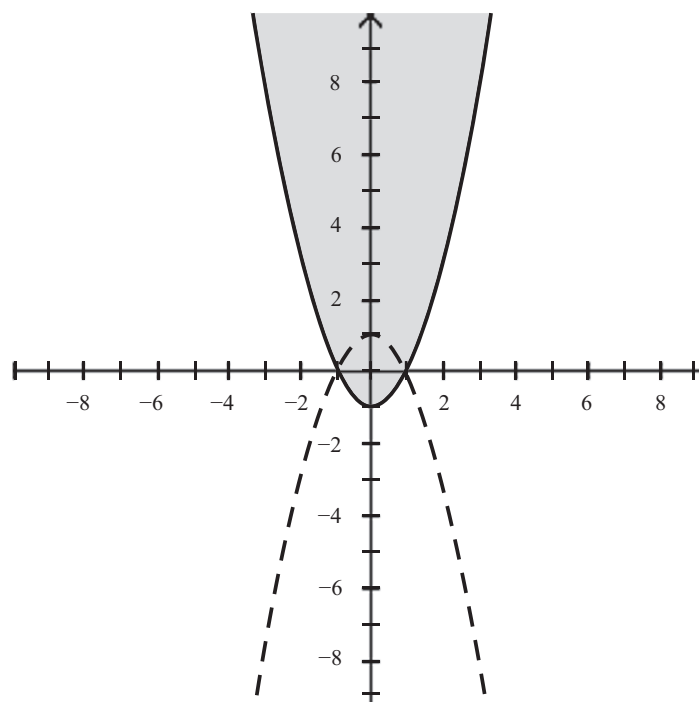
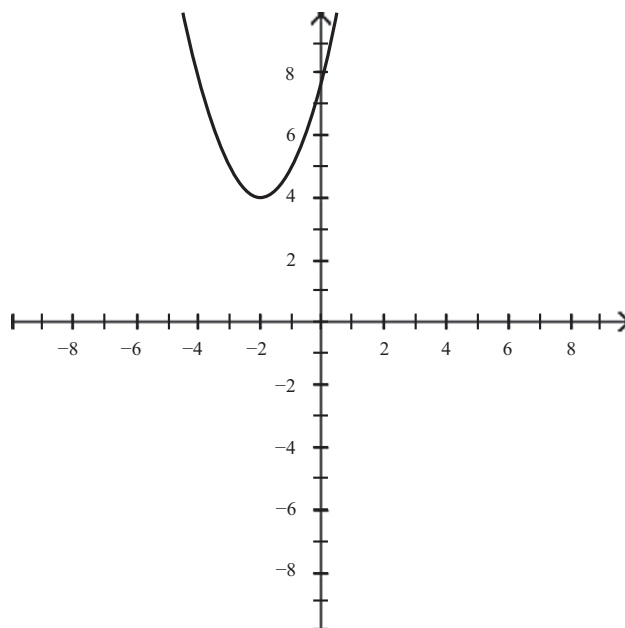We then shade in the solution set of the second inequality.
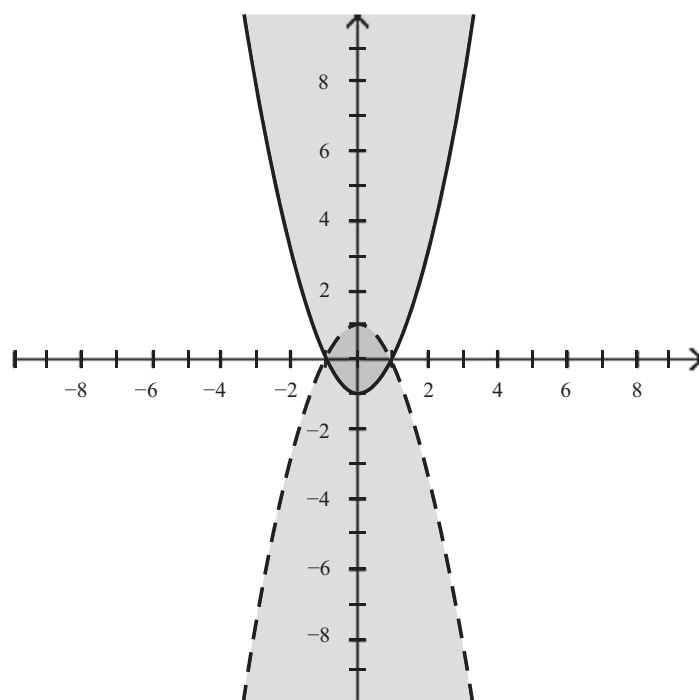
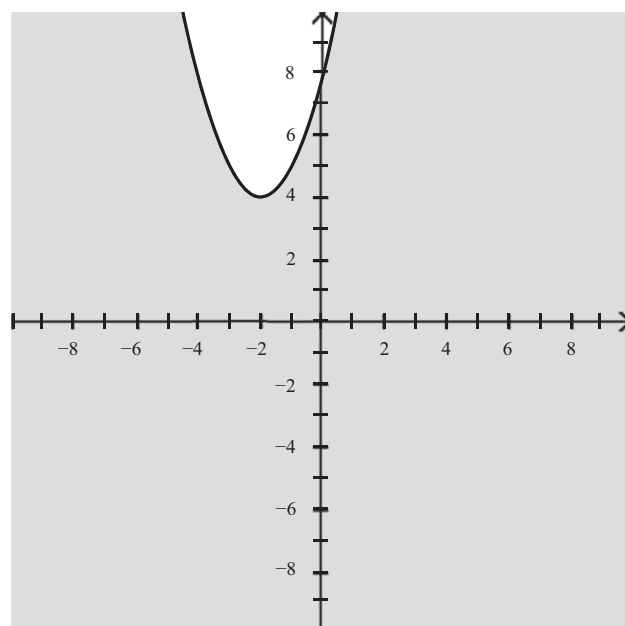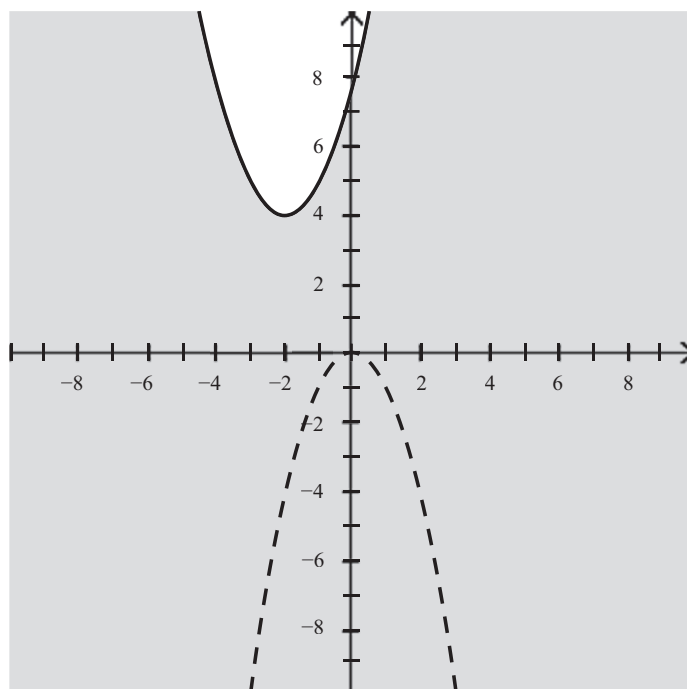The final answer is the intersection of these 2 regions.
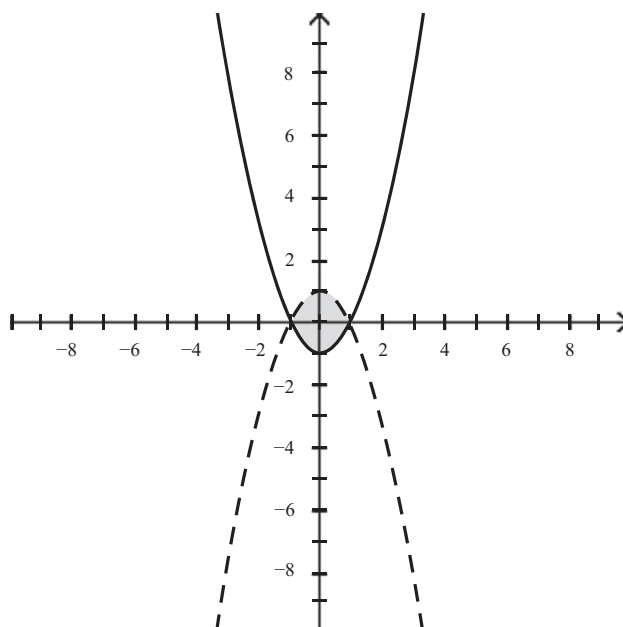
7. $y \geq x^2 + 3$
       $y \geq x^2 - 2x + 1$

We sketch the graph of the parabola that serves as the boundary of the first inequality.
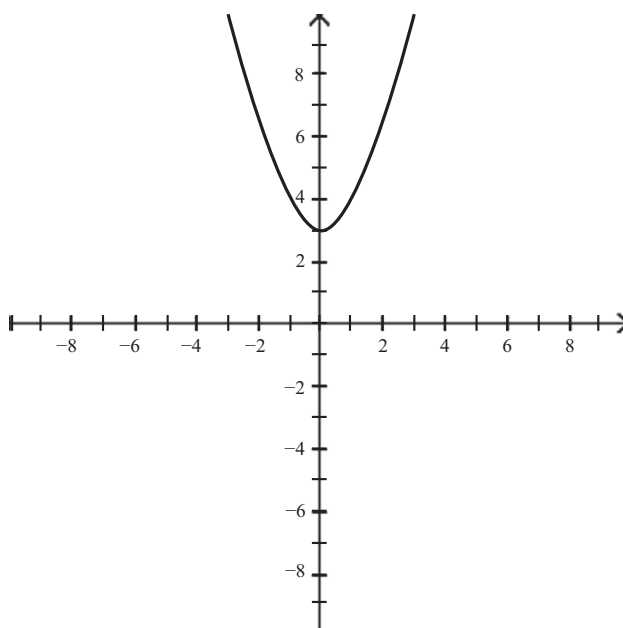
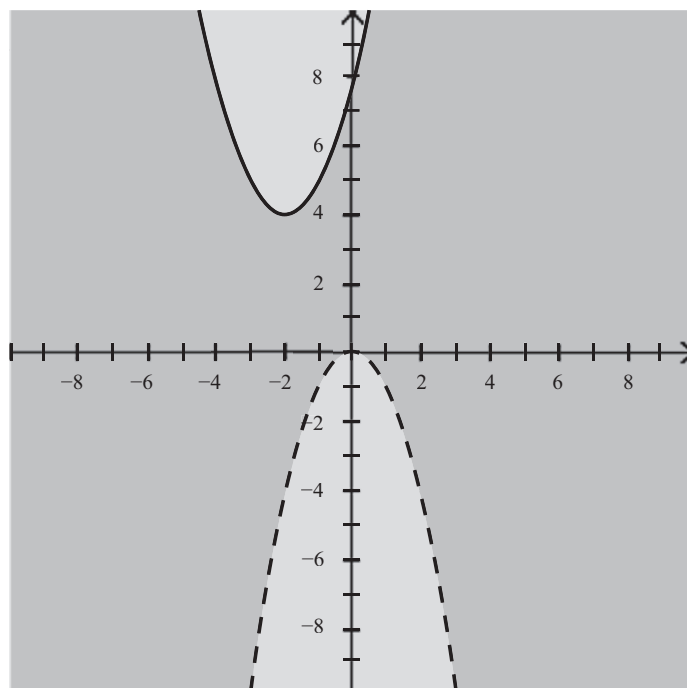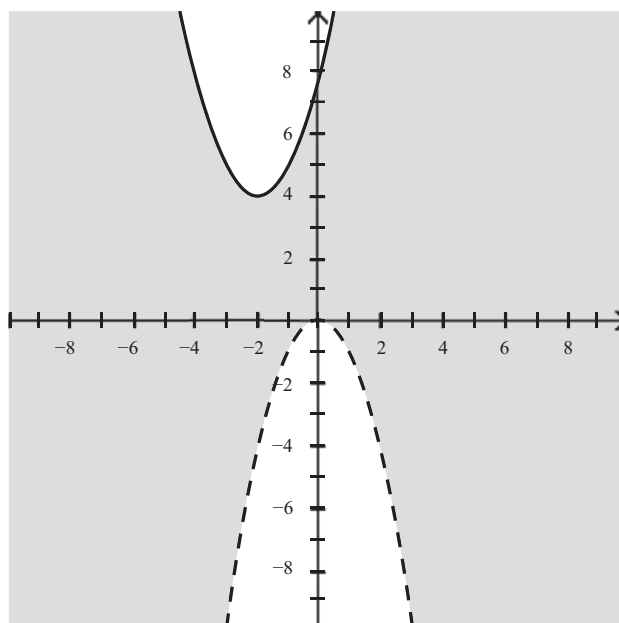We then shade in the solution set of the first inequality.
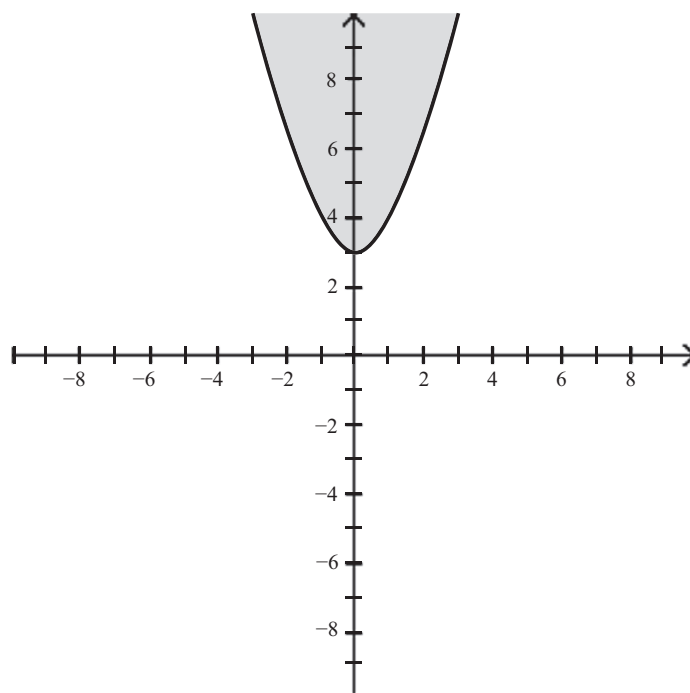
Next, we sketch the graph of the parabola that serves as the boundary of the second inequality.
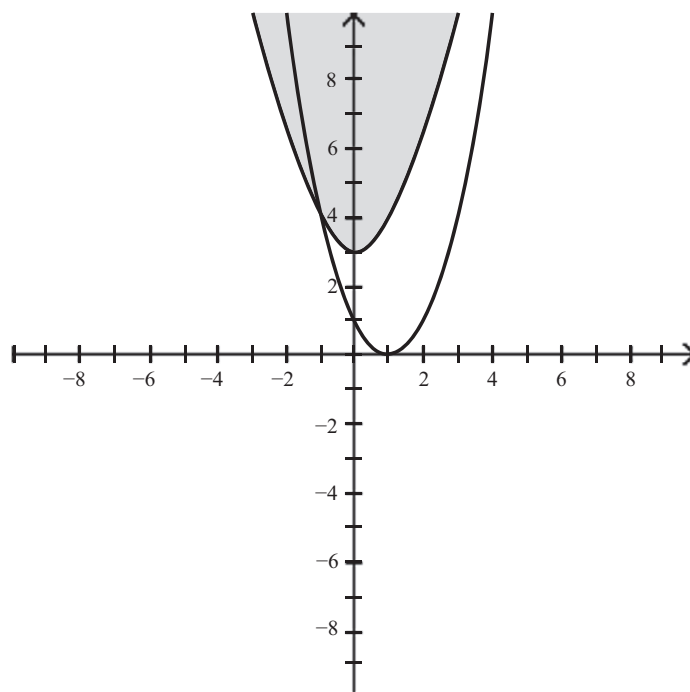
We then shade in the solution set of the second inequality.
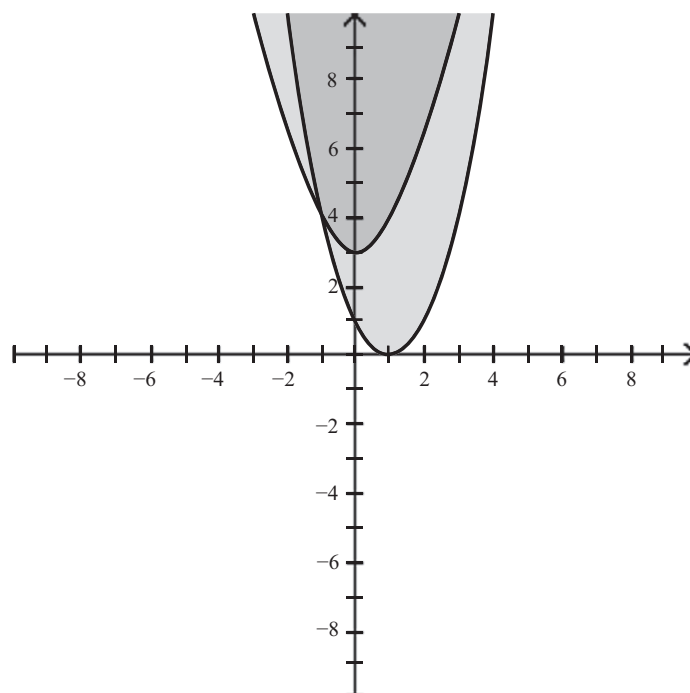
The final answer is the intersection of these 2 regions.
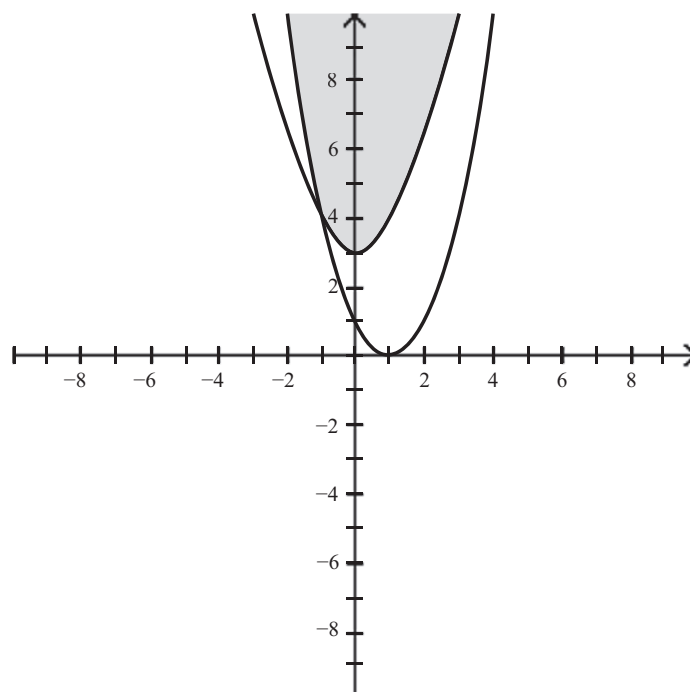
8. $y \le (x+2)^2 + 4$

$y > -x^2$

We sketch the graph of the parabola that serves as the boundary of the first inequality.

We then shade in the solution set of the first inequality.

Next, we sketch the graph of the parabola that serves as the boundary of the second inequality.

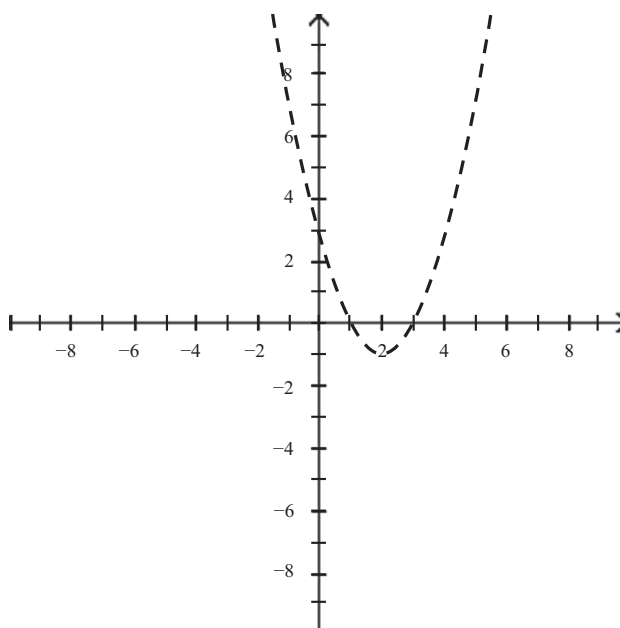We then shade in the solution set of the second inequality.

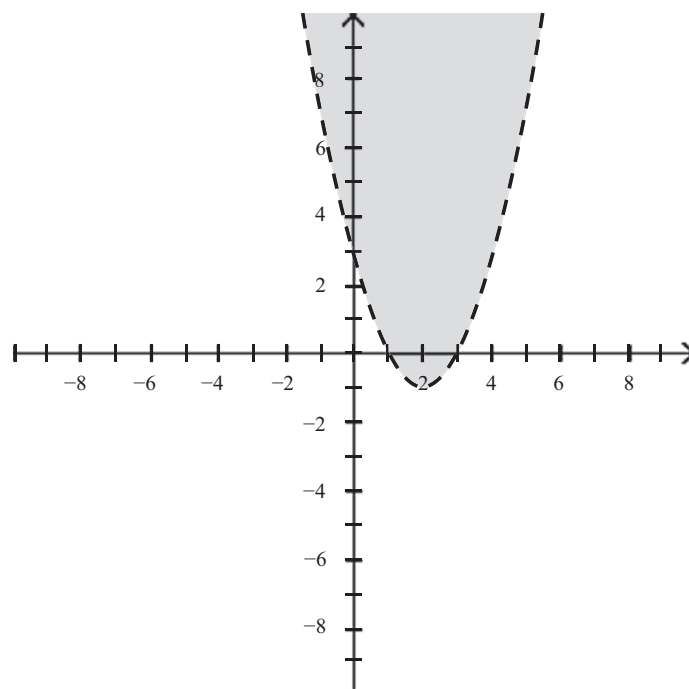The final answer is the intersection of these 2 regions.

9. $y > x^2 - 4x + 3$
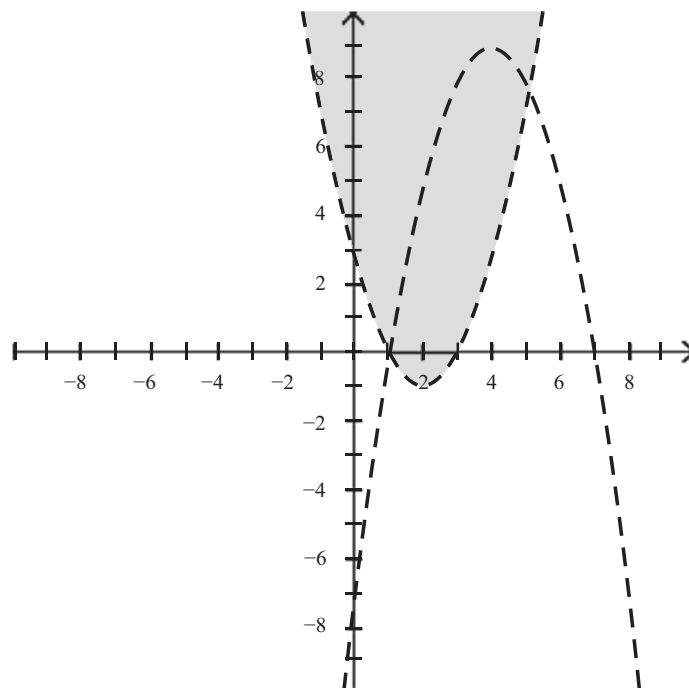 $y < -x^2 + 8x - 7$

We sketch the graph of the parabola that serves as the boundary of the first inequality.
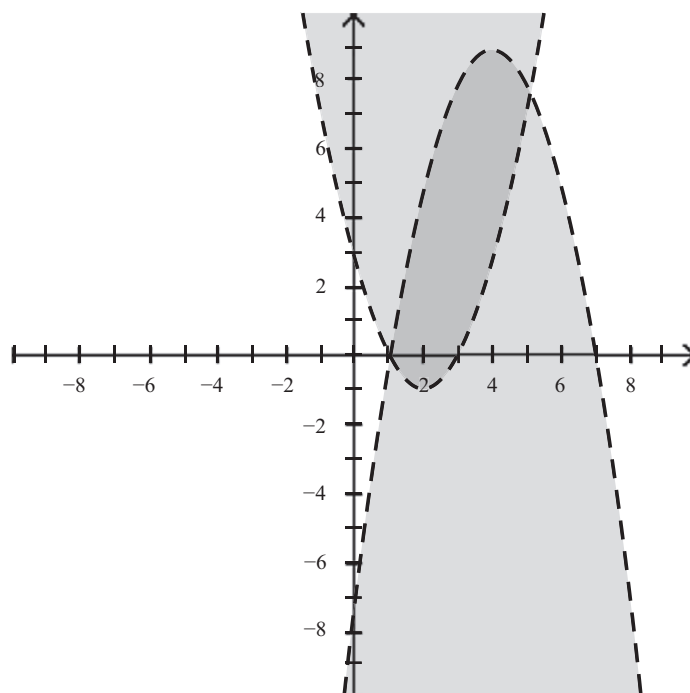
We then shade in the solution set of the first inequality.

Next, we sketch the graph of the parabola that serves as the boundary of the second inequality.

We then shade in the solution set of the second inequality.

The final answer is the intersection of these 2 regions.

10. $y \geq x^2$
            $y > (x-3)^2$

We sketch the graph of the parabola that serves as the boundary of the first inequality.

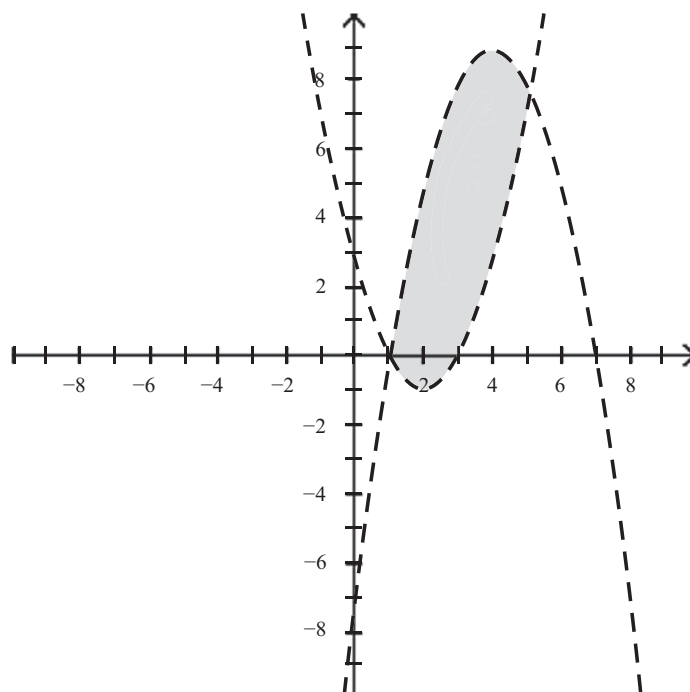We then shade in the solution set of the first inequality.

Next, we sketch the graph of the parabola that serves as the boundary of the second inequality.

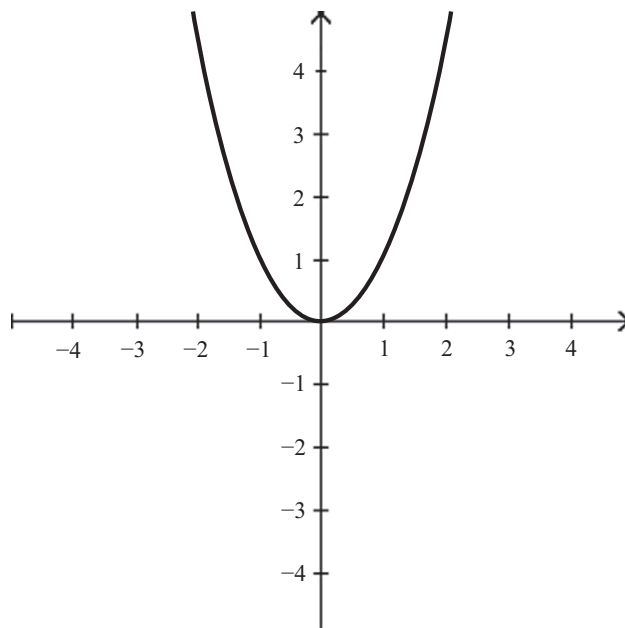We then shade in the solution set of the second inequality.

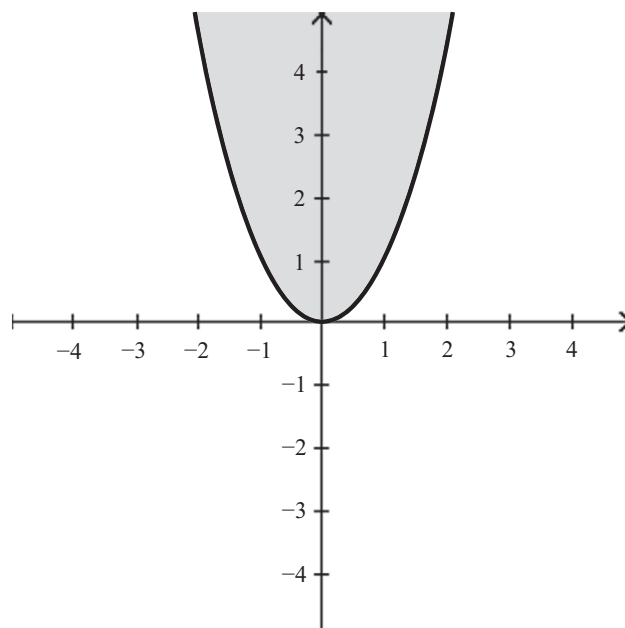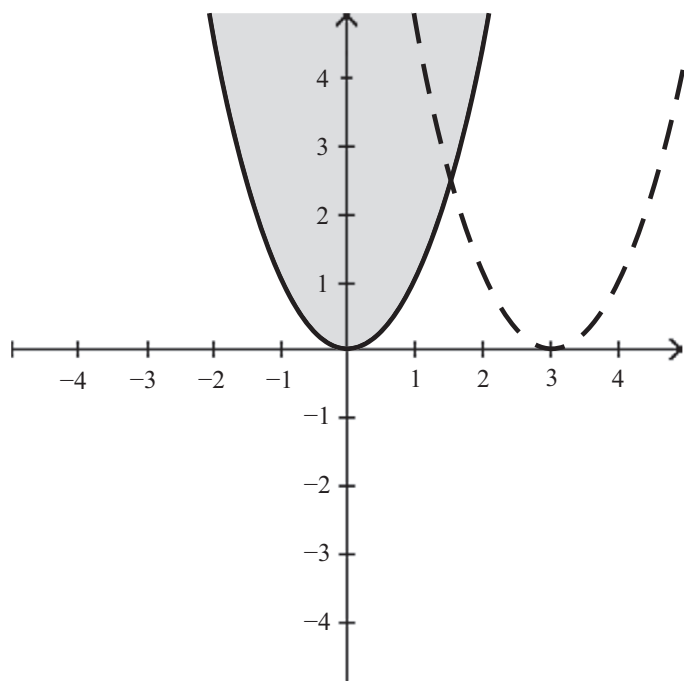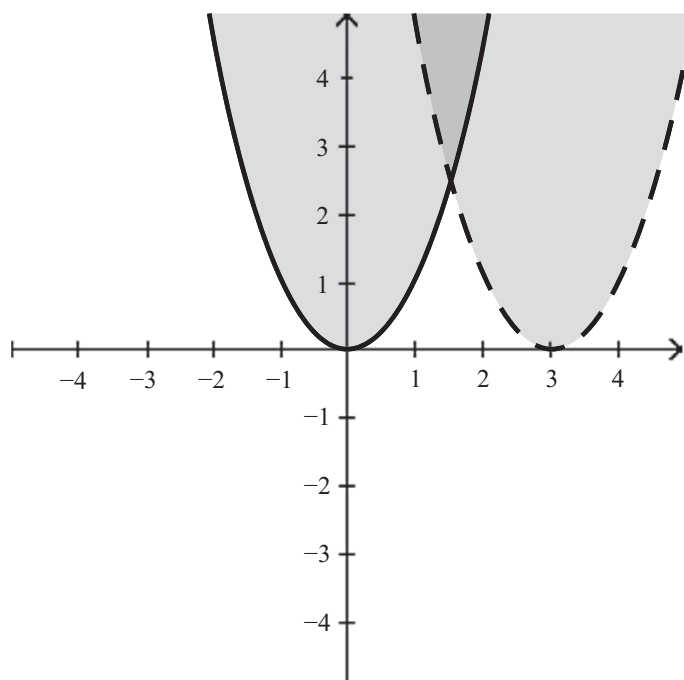The final answer is the intersection of these 2 regions.
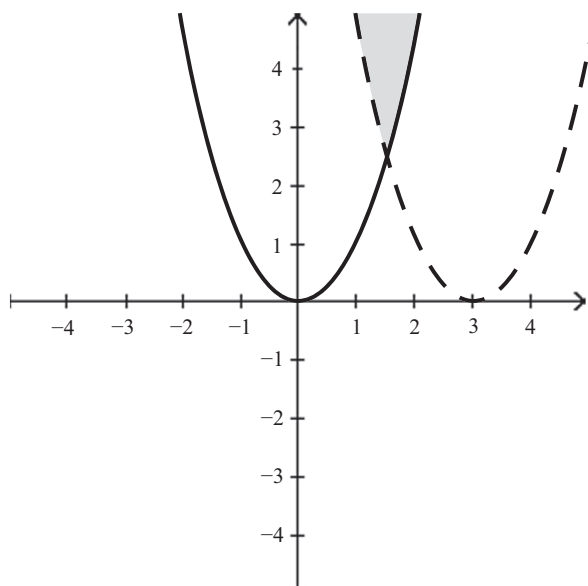
Lesson 17

1. $x^2 + 5y = 7x$

This equation corresponds to a parabola because of the presence of an x^2 term but no y^2 term.

2. $3x^2 - 4y^2 = 60$

This equation corresponds to a hyperbola. This is especially clear once we divide both sides by 60 to get the equation into standard form.

3. $2y^2 = 4x^2 + 100$

This equation corresponds to a hyperbola. Notice that it can be put in standard form as follows.

$$2y^2 = 4x^2 + 100$$
$$2y^2 - 4x^2 = 100$$
$$\frac{2y^2}{100} - \frac{4x^2}{100} = 1$$
$$\frac{y^2}{50} - \frac{x^2}{25} = 1$$

4.

$$\frac{x^2}{16} - \frac{y^2}{49} = 1$$

This hyperbola will open along the *x*-axis. The vertices are located at (4, 0) and (−4, 0).

5.

$$\frac{y^2}{9} - \frac{x^2}{64} = 1$$

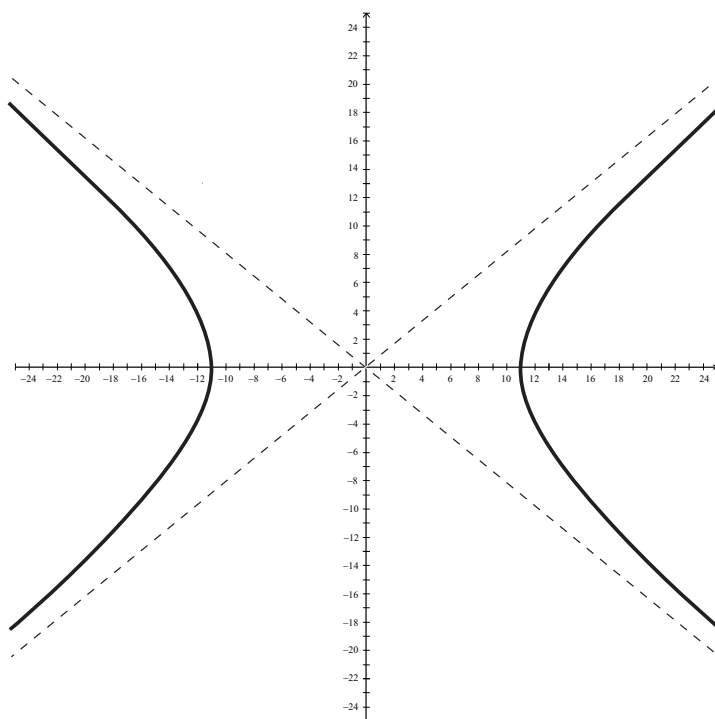This hyperbola will open along the *y*-axis. The vertices are located at (0, 3) and (0, −3).

6.

$$\frac{y^2}{81} - \frac{x^2}{25} = 1$$

This hyperbola will open along the *y*-axis. The vertices are located at (0, 9) and (0, −9).

7.

$$\frac{x^2}{121} - \frac{y^2}{81} = 1$$

8. $\dfrac{x^2}{16} - \dfrac{y^2}{49} = 1$
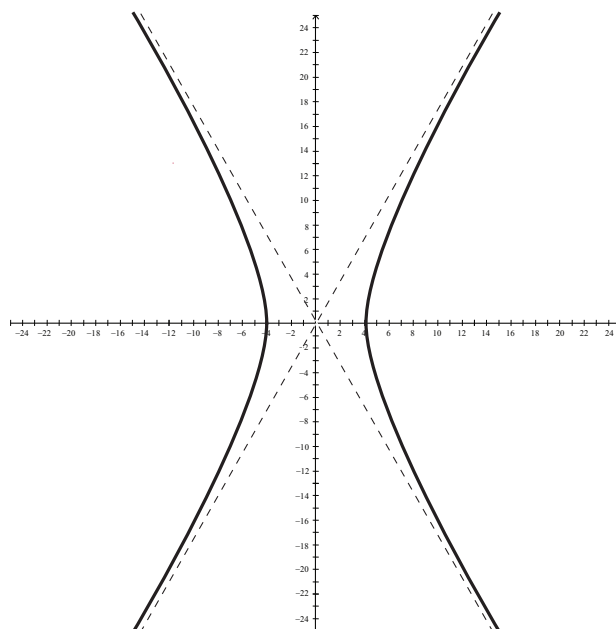
9. $y^2 - \dfrac{x^2}{16} = 1$

This equation is the same as $\dfrac{y^2}{1} - \dfrac{x^2}{16} = 1$.
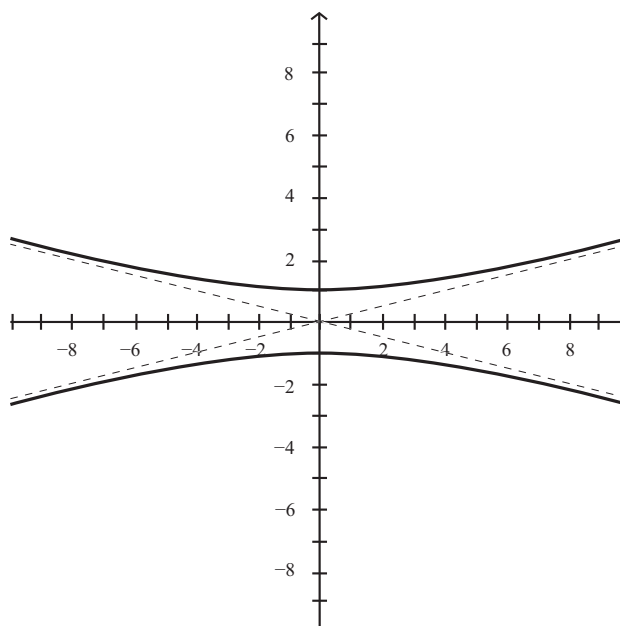
10. $\dfrac{y^2}{64} - \dfrac{x^2}{9} = 1$
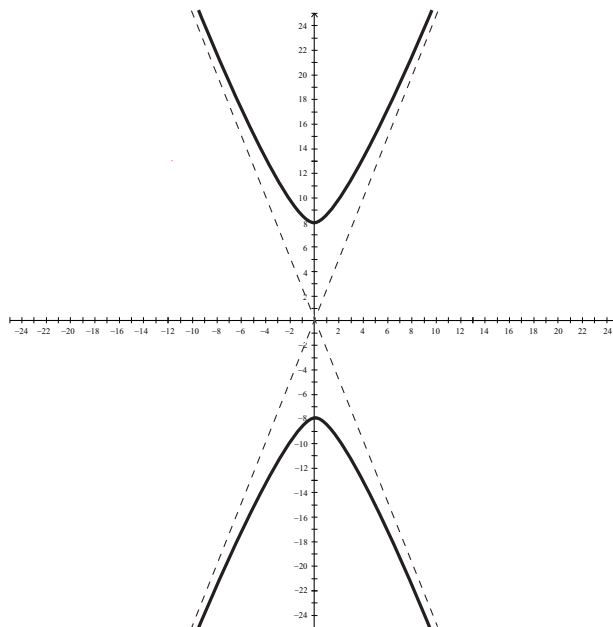
Lesson 18

1. $\dfrac{x^2}{100} + \dfrac{y^2}{121} = 1$

This is the equation of an ellipse.

2. $\dfrac{x^2}{100} + \dfrac{y^2}{100} = 1$

This is the equation of a circle because the denominators of both fractions on the left-hand side of the equation are equal.

3. $6x^2 + 7y^2 = 42$

If we divide both sides of this equation by 42 we have $\dfrac{x^2}{7} + \dfrac{y^2}{6} = 1$. This is the standard equation of an ellipse.

4. $\dfrac{3}{13}$

5. $\dfrac{8}{9}$

6. Since this equation can also be written as $\dfrac{x^2}{1} + \dfrac{y^2}{25} = 1$, the eccentricity of this ellipse is $\dfrac{1}{5}$.

7. $\dfrac{x^2}{9} + \dfrac{y^2}{225} = 1$
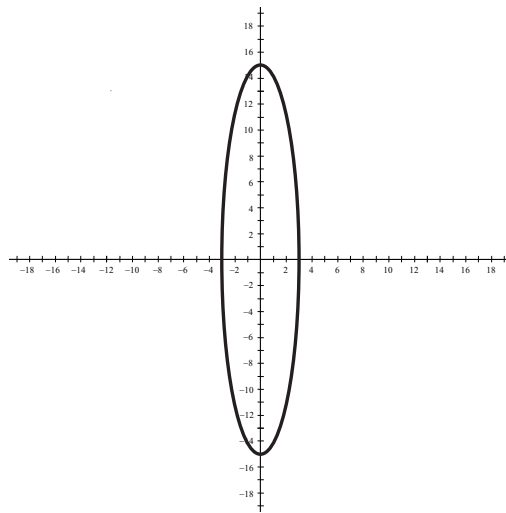
8. $\dfrac{x^2}{225} + \dfrac{y^2}{9} = 1$
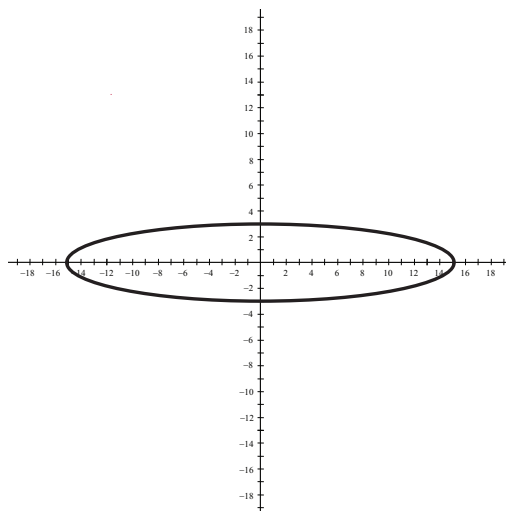
9. $\dfrac{x^2}{25} + \dfrac{y^2}{25} = 1$
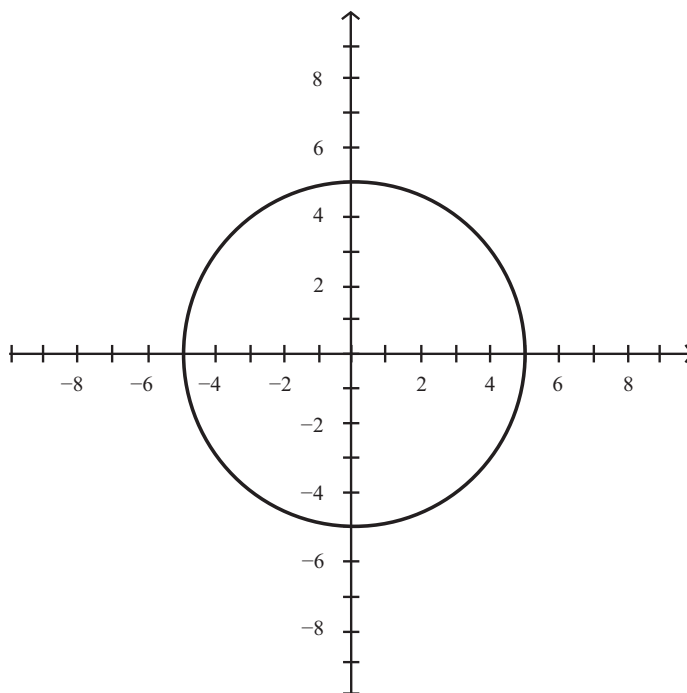
10. $25x^2 + 16y^2 = 400$
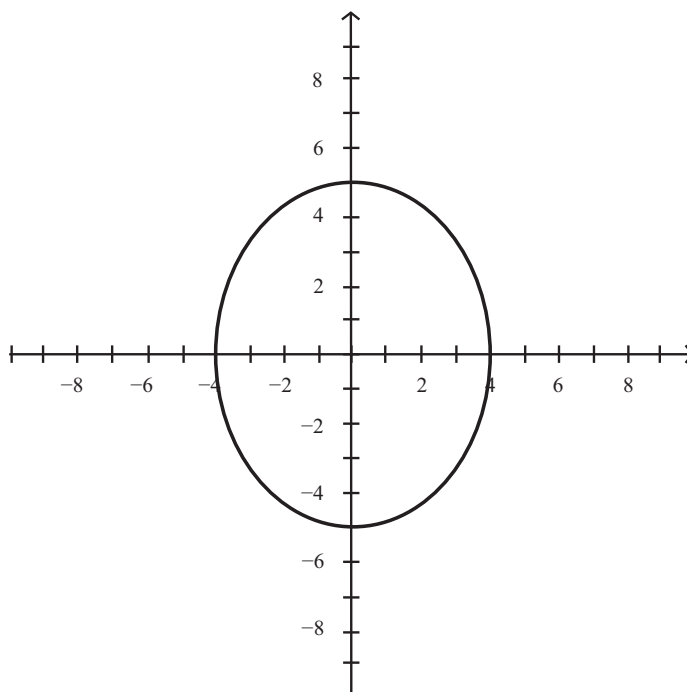
1. Although the terms under the square root symbol make up a polynomial, $\sqrt{x^2+1}$ is not a polynomial.

2. This is a polynomial. Note that it can be rewritten as below, in the standard form of a polynomial.

    $$(x-2)^2 + 4x - 1$$
    $$= x^2 - 4x + 4 + 4x - 1$$
    $$= x^2 + 3$$

3. This is a polynomial.

4. This is not a polynomial because of the negative exponents.

5. Cubic trinomial.

6. Quintic trinomial.

7. Binomial of degree 10.

8. $f(x) = 6x^2 + 12x - 3$

    $f(0) = 6(0)^2 + 12(0) - 3$
    $f(0) = 6(0) + 12(0) - 3$
    $f(0) = 0 + 0 - 3$
    $f(0) = -3$

    $f(1) = 6(1)^2 + 12(1) - 3$
    $f(1) = 6(1) + 12(1) - 3$
    $f(1) = 6 + 12 - 3$
    $f(1) = 15$

9. $f(x) = -x^4 + 2x + 13$

$f(0) = -(0)^4 + 2(0) + 13$
$f(0) = -0 + 2(0) + 13$
$f(0) = -0 + 0 + 13$
$f(0) = 13$

$f(1) = -(1)^4 + 2(1) + 13$
$f(1) = -1 + 2(1) + 13$
$f(1) = -1 + 2 + 13$
$f(1) = 14$

10. $f(x) = x^5 + x^4 + x^3 + x^2 + x$

$f(0) = (0)^5 + (0)^4 + (0)^3 + (0)^2 + 0$
$f(0) = 0 + 0 + 0 + 0 + 0$
$f(0) = 0$

$f(1) = (1)^5 + (1)^4 + (1)^3 + (1)^2 + 1$
$f(1) = 1 + 1 + 1 + 1 + 1$
$f(1) = 5$

Lesson 20

1.   This is not the graph of a polynomial because of the sharp corner at the origin.

2.   This is not the graph of a polynomial because of the break that occurs in the graph.

3.   This is not the graph of a polynomial because of the break that occurs in the graph.

4.   This is not the graph of a polynomial because the domain does not appear to be the set of all real numbers.

Solutions

5.   Both ends of this graph will go up because the degree is even and the leading coefficient is positive. Here is a sketch of that graph.
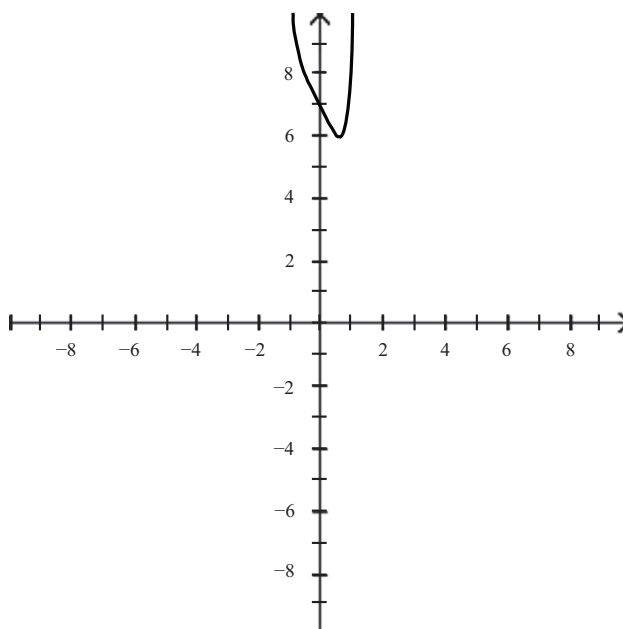
6.   Since the degree of the polynomial is odd, we know one of the ends of the graph goes up while the other goes down. Since the leading coefficient is negative, we know that the end on the left-hand side of the graph goes up while the end on the right-hand side of the graph goes down. Here is a sketch of that graph.
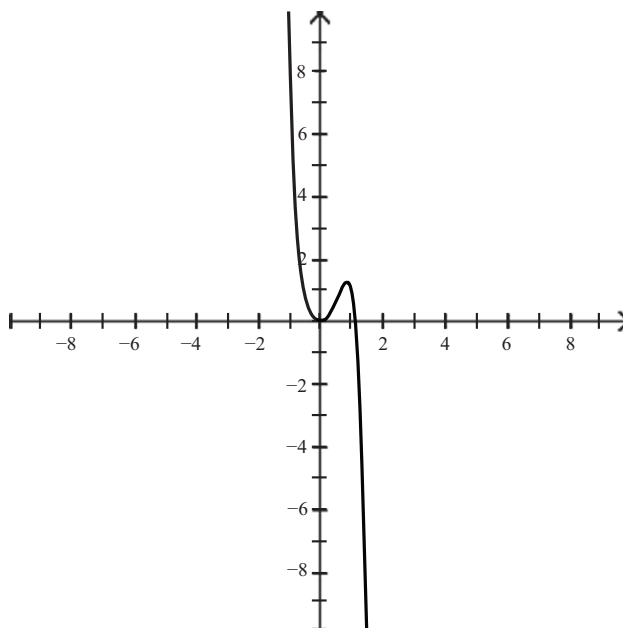
7. Both ends of this graph will go down because the degree is even and the leading coefficient is negative. Here is a sketch of that graph.
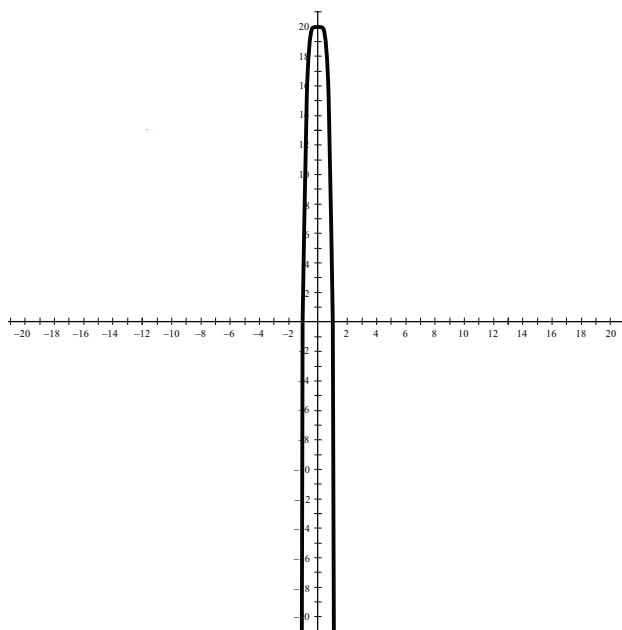
8. We see several things here. First, if we multiplied the function completely, we would have a quartic polynomial, so the degree is even. Moreover, the coefficient in front of the x^4 term would be -1, so the 2 ends of the graph must be pointing down. Also, thanks to the factored version of the function, we already know that the x-intercepts are $(0, 0)$, $(2, 0)$, $(4, 0)$, and $(6, 0)$. If we plot a few more points and connect the dots, we see that the graph is the following.
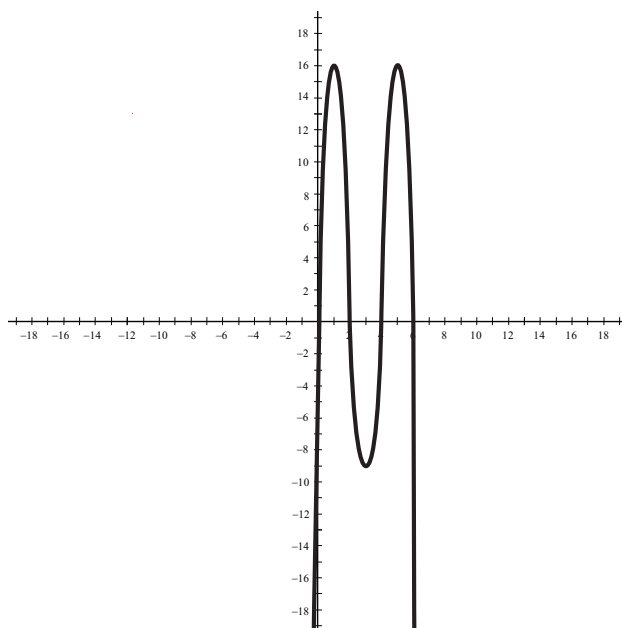
9.   We see that this function will factor.

$$f(x) = x^3 - x$$
$$f(x) = x(x^2 - 1)$$
$$f(x) = x(x-1)(x+1)$$

So we know the 3 x-intercepts of this graph: (0, 0), (1, 0), and (−1, 0). Also, we know the end behavior since the degree is odd and the leading coefficient is positive. This means the left-hand end of the graph will go down while the right-hand end of the graph will go up. Plotting a few other points and connecting the dots gives us the following.
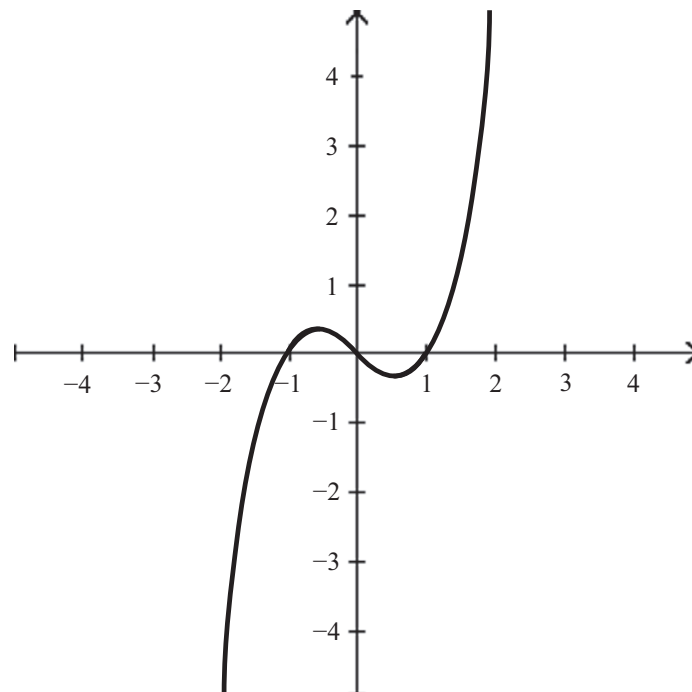
10.   We see that this function is already partly factored, so we can complete the factorization as below.

$$f(x) = (x^2 - 1)^2$$
$$f(x) = ((x-1)(x+1))^2$$
$$f(x) = (x-1)^2 (x+1)^2$$

This tells us at least 2 things. First of all, we know that the *x*-intercepts are at the points (−1, 0) and (1, 0), and we know that these are the only *x*-intercepts. Second, we know that the range of this function never includes any negative numbers—notice that the function began as the square of another quantity. So the graph of this function can never go below the *x*-axis. We also know that the degree of this function is 4 and that the leading coefficient is 1 (if we multiply everything out). So the ends of this graph will both go up. Plotting several additional points and connecting the dots gives us the following.
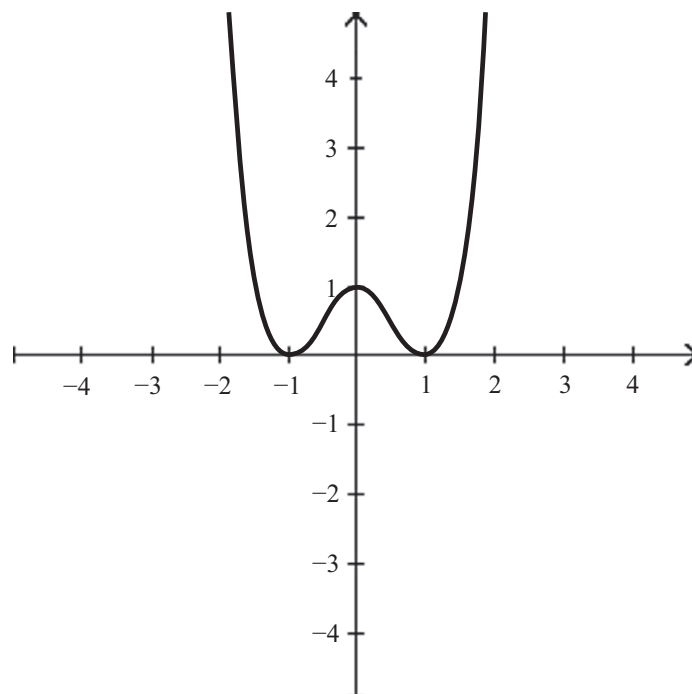
Lesson 21

1. $(x^5 - 2x^3 + 4x) + (x^3 + 5x^2 - 6x + 7)$
    $= x^5 - x^3 + 5x^2 - 2x + 7$

2. $(x^6 - 2x^4 + 3x^2 + 10) - (x^3 + 5x^2 - 6x + 7)$
    $= x^6 - 2x^4 + 3x^2 + 10 - x^3 - 5x^2 + 6x - 7$
    $= x^6 - 2x^4 - x^3 - 2x^2 + 6x + 3$

3. $(x^4 - 1)(x^2 + 2x + 1)$
    $= x^6 + 2x^5 + x^4 - x^2 - 2x - 1$

Solutions

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4. $(x^2 + 3x + 2)(2x^2 - 5x - 6)$

$= 2x^4 - 5x^3 - 6x^2 + 6x^3 - 15x^2 - 18x + 4x^2 - 10x - 12$

$= 2x^4 + x^3 - 17x^2 - 28x - 12$

5.

$$
\begin{array}{r}
x - 4 \\
x - 3 \overline{)\, x^2 - 7x + 12} \\
\underline{x^2 - 3x} \\
-4x + 12 \\
\underline{-4x + 12} \\
0
\end{array}
$$

The answer is $x - 4$.

6.

$$
\begin{array}{r}
x^2 + 3x + 5 \\
x - 2 \overline{)\, x^3 + x^2 - x - 10} \\
\underline{x^3 - 2x^2} \\
3x^2 - x \\
\underline{3x^2 - 6x} \\
5x - 10 \\
\underline{5x - 10} \\
0
\end{array}
$$

The answer is $x^2 + 3x + 5$.

7. $f(x) = 4x + 3$ and $g(x) = 5x - 7$

$f(g(x)) = f(5x - 7) = 4(5x - 7) + 3 = 20x - 28 + 3 = 20x - 25$

$g(f(x)) = g(4x + 3) = 5(4x + 3) - 7 = 20x + 15 - 7 = 20x + 8$

8. $f(x) = x^2 + 1$ and $g(x) = x^2 - 7$

$f(g(x)) = f(x^2 - 7) = (x^2 - 7)^2 + 1 = x^4 - 14x^2 + 49 + 1 = x^4 - 14x^2 + 50$

$g(f(x)) = g(x^2 + 1) = (x^2 + 1)^2 - 7 = x^4 + 2x^2 + 1 - 7 = x^4 + 2x^2 - 6$

9. $f(x) = x^2 + 3x$ and $g(x) = 5x - 1$

$f(g(x)) = f(5x-1) = (5x-1)^2 + 3(5x-1) = 25x^2 - 10x + 1 + 15x - 3 = 25x^2 + 5x - 2$

$g(f(x)) = g(x^2 + 3x) = 5(x^2 + 3x) - 1 = 5x^2 + 15x - 1$

10. $f(x) = x^3 + 3x$ and $g(x) = x^2$

$f(g(x)) = f(x^2) = (x^2)^3 + 3x^2 = x^6 + 3x^2$

$g(f(x)) = g(x^3 + 3x) = (x^3 + 3x)^2 = x^6 + 6x^4 + 9x^2$

Lesson 22

1.
$$x^6 - 12x^5 + 27x^4 = 0$$
$$x^4(x^2 - 12x + 27) = 0$$
$$x^4(x-9)(x-3) = 0$$

So the solutions are $x = 0$, $x = 3$, and $x = 9$.

2.
$$3x^4 - 27x^2 = 0$$
$$3x^2(x^2 - 9) = 0$$
$$3x^2(x-3)(x+3) = 0$$

So the solutions are $x = 0$, $x = 3$, and $x = -3$.

3.
$$2x^5 + 40x^4 + 81x^3 - 5x^2 = x^5 + 21x^4 - 3x^3 - 5x^2$$
$$x^5 - 19x^4 + 84x^3 = 0$$
$$x^3(x^2 - 19x + 84) = 0$$
$$x^3(x-12)(x-7) = 0$$

So the solutions are $x = 0$, $x = 7$, and $x = 12$.

4.
$$x^3 - 64 = 0$$
$$(x-4)(x^2 + 4x + 16) = 0$$

Because the discriminant of $x^2 + 4x + 16$ is negative, the only real number solution is $x = 4$.

5.
$$4x^5 - 32x^2 = 0$$
$$4x^2(x^3 - 8) = 0$$
$$4x^2(x-2)(x^2 + 2x + 4) = 0$$

Because the discriminant of $x^2 + 2x + 4$ is negative, the real number solutions are $x = 0$ and $x = 2$.

6.
$$5x^6 + 320x^3 = 0$$
$$5x^3(x^3 + 64) = 0$$
$$5x^3(x+4)(x^2 - 4x + 16) = 0$$

Because the discriminant of $x^2 - 4x + 16$ is negative, the real number solutions are $x = 0$ and $x = -4$.

7.
$$16x^4 + 250x = 0$$
$$2x(8x^3 + 125) = 0$$
$$2x((2x)^3 + 5^3) = 0$$
$$2x(2x+5)(4x^2 - 10x + 25) = 0$$

Because the discriminant of $4x^2 - 10x + 25$ is negative, the only real number solutions are $x = 0$ and $x = -\dfrac{5}{2}$.

8.
$$x^4 - 11x^2 + 28 = 0$$
$$(x^2 - 7)(x^2 - 4) = 0$$

So the real number solutions are $x = -\sqrt{7}, x = \sqrt{7}, x = -2, x = 2$.

9.
$$3x^4 + 20x^2 + 50 = 2x^4 + 5x^2 - 4$$
$$x^4 + 15x^2 + 54 = 0$$
$$(x^2 + 9)(x^2 + 6) = 0$$

Because the discriminant of each of these 2 quadratic terms is negative, this equation has no real number solutions.

10.
$$x^6 - 64 = 0$$
$$(x^3 - 8)(x^3 + 8) = 0$$
$$(x - 2)(x^2 + 2x + 4)(x + 2)(x^2 - 2x + 4) = 0$$

Because the discriminants of $x^2 - 2x + 4$ and $x^2 + 2x + 4$ are negative, the real number solutions are $x = -2$ and $x = 2$.

Lesson 23

1. $\dfrac{1}{1}, -\dfrac{1}{1}, \dfrac{2}{1}, -\dfrac{2}{1}, \dfrac{3}{1}, -\dfrac{3}{1}, \dfrac{4}{1}, -\dfrac{4}{1}, \dfrac{6}{1}, -\dfrac{6}{1}, \dfrac{12}{1}, -\dfrac{12}{1}$; or simply, 1, −1, 2, −2, 3, −3, 4, −4, 6, −6, 12, −12.

2. $\dfrac{1}{1}, -\dfrac{1}{1}, \dfrac{5}{1}, -\dfrac{5}{1}, \dfrac{7}{1}, -\dfrac{7}{1}, \dfrac{35}{1}, -\dfrac{35}{1}$; or simply, 1, −1, 5, −5, 7, −7, 35, −35.

3. $\dfrac{1}{1}, -\dfrac{1}{1}, \dfrac{1}{2}, -\dfrac{1}{2}$; or $1, -1, \dfrac{1}{2}, -\dfrac{1}{2}$.

4. Plugging in the possible roots found in problem 1 above, we find that $x = 3$ is the only value in the list that makes the equation true. So 3 is the only rational root of this equation.

5. Plugging in the possible roots found in problem 2 above, we find that $x = 5$ is the only value in the list that makes the equation true. So 5 is the only rational root of this equation.

6. Plugging in the possible roots found in problem 3 above, we find that $x = 1$, $x = -1$, and $x = -\dfrac{1}{2}$ all make the equation true. So these are all rational roots of the original equation.

7. $x^4 - 8x^3 + 2x^2 + 8x - 3 = 0$

With the rational roots theorem, we can check for rational roots very quickly. We find that $x = 1$ and $x = -1$ are roots. From the factor theorem, we know that $x - 1$ and $x + 1$ are factors of the left-hand side. We can then use long division to divide out $x - 1$ and $x + 1$ from the original left-hand side and see that the equation can be rewritten as below.

$$(x-1)(x+1)(x^2 - 8x + 3) = 0$$

Now the quadratic formula can be brought into play to find the roots coming from the last factor on the left-hand side.

$$x = \frac{8 \pm \sqrt{64 - 12}}{2}$$
$$x = \frac{8 \pm \sqrt{52}}{2}$$
$$x = \frac{8 \pm 2\sqrt{13}}{2}$$
$$x = 4 \pm \sqrt{13}$$

So all the roots are $x = 1$, $x = -1$, and $x = 4 \pm \sqrt{13}$.

8. $x^4 + x^3 - 3x^2 - 4x - 4 = 0$

With the rational roots theorem, we can check for rational roots very quickly. We find that $x = 2$ and $x = -2$ are roots. From the factor theorem, we know $x - 2$ and $x + 2$ are factors of the left-hand side. We can then use long division to divide out $x - 2$ and $x + 2$ from the original left-hand side and see that the equation can be rewritten as below.

$$(x-2)(x+2)(x^2 + x + 1) = 0$$

Since the discriminant of the last factor on the left-hand side is negative, we know that there will be no real roots contributed by this factor. So the only real roots are $x = 2$ and $x = -2$.

9. $6x^4 + 5x^3 + 7x^2 + 5x + 1 = 0$

With the rational roots theorem, we can check for rational roots very quickly. We find that $x = -1/2$ and $x = -1/3$ are roots. From the factor theorem, we know $x + 1/2$ and $x + 1/3$ are factors of the left-hand side. Clearing out denominators, we see that this is the same as saying that $2x + 1$ and $3x + 1$ are factors. We can then use long division to see that the equation can be rewritten as below.

$$(2x+1)(3x+1)(x^2 + 1) = 0$$

Since the discriminant of the last factor on the left-hand side is negative, we know it will not contribute any real roots. So the only real roots are $x = -1/2$ and $x = -1/3$.

10. $2x^4 + 5x^3 + 4x^2 + x = 0$

We begin by factoring out the common term on the left-hand side.

$$x(2x^3 + 5x^2 + 4x + 1) = 0$$

The factor of x in front of the parenthesis on the left-hand side of the equation means that $x = 0$ is a root. Now we can concentrate on $2x^3 + 5x^2 + 4x + 1 = 0$. The rational roots theorem allows us to check for rational roots very quickly. We find that $x = -1/2$ and $x = -1$ are roots. From the factor theorem, we know $x + 1/2$ and $x + 1$ are factors of the left-hand side. Clearing out denominators, we see that this is the same as saying that $2x + 1$ and $x + 1$ are factors. We can then use long division to see that the equation can be rewritten as below.

$$(2x + 1)(x + 1)^2 = 0$$

So the only real roots are $x = 0$, $x = -\dfrac{1}{2}$, and $x = -1$.

Lesson 24

1.   The sign changes in the left-side polynomial are $++, +-$. There is 1 sign change, which implies that the number of positive real solutions is 1. When x is replaced by $-x$, the sign changes are $+-, --$. So there is 1 sign change, which implies that the number of negative real solutions is 1.

2.   The sign changes in the left-side polynomial are $+-, -+$. There are 2 sign changes, which implies that the number of positive real solutions is 2 or 0. When x is replaced by $-x$, the sign changes are $+-, --$. So there is 1 sign change, which implies that the number of negative real solutions is 1.

3.   The sign changes in the left-side polynomial are $++, ++, ++, +-, --$. There is 1 sign change, which implies that the number of positive real solutions is 1. When x is replaced by $-x$, the sign changes are $-+, +-, -+, ++, +-$. So there are 4 sign changes, which implies that the number of negative real solutions is 4, 2, or 0.

4.   The sign changes in the left-side polynomial are $+-, --, -+, ++, +-$. There are 3 sign changes, which implies that the number of positive real solutions is 3 or 1. When x is replaced by $-x$, the sign changes are $--, -+, ++, +-, --$. So there are 2 sign changes, which implies that the number of negative real solutions is 2 or 0.

5.   The sign changes in the left-side polynomial are $+-, -+$. There are 2 sign changes, which implies that the number of positive real solutions is 2 or 0. When x is replaced by $-x$, the sign changes are $+-, -+$. So there are 2 sign changes, which implies that the number of negative real solutions is 2 or 0.

6. $x^3 - 11x^2 - 24x - 10 = x + 3$

We begin by placing all the terms on the left-hand side of the equation.

$$x^3 - 11x^2 - 24x - 10 = x + 3$$
$$x^3 - 11x^2 - 25x - 13 = 0$$

We see that there is 1 sign change of the form $+ -$ that occurs at the beginning of the polynomial on the left-hand side. Because of Descartes' rule of signs, we know that there must be exactly 1 positive real root. When we replace x by $-x$ in this equation, there are 2 sign changes, so there must be either 2 or 0 negative real roots.

If the one positive real root is a rational root, then it must be either 1 or 13. So we check both of these to see whether they are roots.

$$1^3 - 11(1)^2 - 25(1) - 13 = 0$$
$$1 - 11 - 25 - 13 = 0$$
$$-48 = 0$$

We find that 1 is not the positive real root.

$$13^3 - 11(13)^2 - 25(13) - 13 = 0$$
$$2197 - 1859 - 325 - 13 = 0$$
$$0 = 0$$

So $x = 13$ is the positive real root that Descartes' rule guaranteed.

Now we factor $x - 13$ from the left-hand side, and we see that the equation can also be written as the below.

$$(x - 13)(x^2 + 2x + 1) = 0$$
$$(x - 13)(x + 1)^2 = 0$$

So we see that the negative real roots are actually $x = -1$ and $x = -1$ (in other words, -1 is repeated twice).

Since the polynomial in question is cubic, there can be no more than 3 roots total (we know this because of the fundamental theorem of algebra). So 13 and -1 are the only roots.

7. $x^4 = 6x^2 - 8$

We start by moving all terms to the left.

$$x^4 = 6x^2 - 8$$
$$x^4 - 6x^2 + 8 = 0$$

Descartes' rule of signs says that the number of possible positive real roots is either 2 or 0 and that the number of possible negative real roots is also 2 or 0. We also know that the potential rational roots are simply 1, −1, 2, −2, 4, −4, 8, and −8. Checking each of these by evaluating the polynomial on the left-hand side by each of these numbers shows us that −2 and 2 are solutions. Thus, we can factor $x + 2$ and $x − 2$ from the left-hand side. Once we do this, we see that we have the following.

$$x^4 - 6x^2 + 8 = 0$$
$$(x + 2)(x - 2)(x^2 - 2) = 0$$

The solutions that arise from the factor $x^2 - 2$ are just $\sqrt{2}$ and $-\sqrt{2}$. Since the left-hand side is quartic, we know that there are at most 4 solutions, and we now know all 4.

8. $x^3 - x^2 + x - 1 = 0$

From Descartes' rule of signs, we know that there are either 3 or 1 positive real solutions and 0 negative real solutions. If the positive real solution is a rational number, then it must be 1. So we try this first.

$$x^3 - x^2 + x - 1 = 0$$
$$(1)^3 - (1)^2 + 1 - 1 = 0$$
$$1 - 1 + 1 - 1 = 0$$
$$0 = 0$$

Since this is true, we know that $x = 1$ is a solution. So $x − 1$ is a factor of the left-hand side.

$$x^3 - x^2 + x - 1 = 0$$
$$(x - 1)(x^2 + 1) = 0$$

Now we simply need to determine the solutions that arise from $x^2 + 1 = 0$. This can be done with the quadratic formula. They are $x = i$ and $x = -i$.

9. $x^4 = 256$

This equation is equivalent to $x^4 - 256 = 0$. Descartes' rule of signs tells us that there must be exactly 1 positive real solution and exactly 1 negative real solution. In fact, we can determine relatively quickly that these are $x = 4$ and $x = - 4$. (For example, we can try the rational numbers 1, 2, 4, 8, 16, 32, 64, 128, and 256, along with their negatives.) This means that $x - 4$ and $x + 4$ are factors of this equation. So we factor as follows.

$$x^4 - 256 = 0$$
$$(x - 4)(x + 4)(x^2 + 16) = 0$$

We now simply find the solutions of $x^2 + 16 = 0$. Thanks to the quadratic formula, we find that they are $x = 4i$ and $x = -4i$. The fundamental theorem of algebra implies that we are done, since the original polynomial was quartic and we have identified 4 solutions.

10. $x^6 + 4x^4 = x^2 + 4$

We begin by moving all terms to the left.

$$x^6 + 4x^4 = x^2 + 4$$
$$x^6 + 4x^4 - x^2 - 4 = 0$$

Descartes' rule of signs tells us that there must be 1 positive real solution and 1 negative real solution. If these real roots are rational, they must be 1, 2, 4, or their negatives. After trying these, we find that 1 and −1 are solutions. So $x - 1$ and $x + 1$ are factors.

$$x^6 + 4x^4 - x^2 - 4 = 0$$
$$(x-1)(x+1)(x^4 + 5x^2 + 4) = 0$$

Now we must determine the solutions that arise from $x^4 + 5x^2 + 4 = 0$. Fortunately, this can be factored.

$$x^4 + 5x^2 + 4 = 0$$
$$(x^2 + 4)(x^2 + 1) = 0$$

We can break this into 2 quadratic equations, and we find that their solutions are (respectively) $x = 2i$, $x = -2i$, $x = i$, and $x = -i$. These are our 6 solutions.

Lesson 25

1. $\sqrt[3]{-500} = \sqrt[3]{-125 \cdot 4} = -5\sqrt[3]{4}$

2. $\sqrt[4]{405}\sqrt[4]{250} = \sqrt[4]{81 \cdot 5}\sqrt[4]{125 \cdot 2} = 3\sqrt[4]{5}\sqrt[4]{125 \cdot 2} = 3\sqrt[4]{625}\sqrt[4]{2} = 3(5)\sqrt[4]{2} = 15\sqrt[4]{2}$

3. $\sqrt[5]{8}\sqrt[5]{12} = \sqrt[5]{8}\sqrt[5]{4 \cdot 3} = \sqrt[5]{32 \cdot 3} = 2\sqrt[5]{3}$

4. $\dfrac{\sqrt[3]{648}}{\sqrt[3]{81}} = \sqrt[3]{\dfrac{648}{81}} = \sqrt[3]{8} = 2$

5.

$$10\sqrt[5]{32} + 2\sqrt[5]{-243} - 5\sqrt[5]{-1}$$
$$= 10(2) + 2(-3) - 5(-1)$$
$$= 20 - 6 + 5$$
$$= 19$$

6.

$$\sqrt[4]{32} - 2\sqrt[4]{4802} - 3\sqrt[4]{512}$$
$$= \sqrt[4]{16 \cdot 2} - 2\sqrt[4]{2401 \cdot 2} - 3\sqrt[4]{256 \cdot 2}$$
$$= 2\sqrt[4]{2} - 2(7)\sqrt[4]{2} - 3(4)\sqrt[4]{2}$$
$$= 2\sqrt[4]{2} - 14\sqrt[4]{2} - 12\sqrt[4]{2}$$
$$= -24\sqrt[4]{2}$$

7.

$$\frac{1}{\sqrt{5} + \sqrt{2}}$$
$$= \frac{1}{\sqrt{5} + \sqrt{2}} \cdot \frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} - \sqrt{2}}$$
$$= \frac{\sqrt{5} - \sqrt{2}}{5 - 2}$$
$$= \frac{\sqrt{5} - \sqrt{2}}{3}$$

8.

$$\frac{6}{\sqrt{31} - \sqrt{30}}$$
$$= \frac{6}{\sqrt{31} - \sqrt{30}} \cdot \frac{\sqrt{31} + \sqrt{30}}{\sqrt{31} + \sqrt{30}}$$
$$= \frac{6(\sqrt{31} + \sqrt{30})}{1}$$
$$= 6(\sqrt{31} + \sqrt{30})$$

9.

$$\frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}}$$
$$= \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}} \cdot \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} - \sqrt{3}}$$
$$= \frac{5 - \sqrt{15} - \sqrt{15} + 3}{5 - 3}$$
$$= \frac{8 - 2\sqrt{15}}{2}$$
$$= 4 - \sqrt{15}$$

10.

$$\frac{5\sqrt{10}}{2\sqrt{10}-4\sqrt{7}}$$

$$=\frac{5\sqrt{10}}{2\sqrt{10}-4\sqrt{7}}\cdot\frac{2\sqrt{10}+4\sqrt{7}}{2\sqrt{10}+4\sqrt{7}}$$

$$=\frac{10\sqrt{100}+20\sqrt{70}}{4\cdot10-16\cdot7}$$

$$=\frac{10\cdot10+20\sqrt{70}}{40-112}$$

$$=\frac{100+20\sqrt{70}}{-72}$$

$$=-\frac{25+5\sqrt{70}}{18}$$

Lesson 26

1.

$$\sqrt[4]{1-5x}=\sqrt[4]{5-x}$$
$$1-5x=5-x$$
$$-4x=4$$
$$x=-1$$

Check it.

$$\sqrt[4]{1-5(-1)}=\sqrt[4]{5-(-1)}$$
$$\sqrt[4]{1+5}=\sqrt[4]{5+1}$$
$$\sqrt[4]{6}=\sqrt[4]{6}$$

So $x=-1$ really is a solution.

2.

$$\sqrt[5]{3x-10}=\sqrt[5]{2-x}$$
$$3x-10=2-x$$
$$4x=12$$
$$x=3$$

Check it.

$$\sqrt[5]{3(3)-10}=\sqrt[5]{2-3}$$
$$\sqrt[5]{9-10}=\sqrt[5]{2-3}$$
$$\sqrt[5]{-1}=\sqrt[5]{-1}$$
$$-1=-1$$

So $x=3$ really is a solution.

3.
$$\sqrt{90-x} = x$$
$$90-x = x^2$$
$$x^2 + x - 90 = 0$$
$$(x+10)(x-9) = 0$$

So $x = -10$ and $x = 9$ are our potential solutions.

Check them.

$$\sqrt{90-(-10)} = -10$$
$$\sqrt{100} = -10$$
$$10 = -10$$

This is false, so $x = -10$ is not a solution.

$$\sqrt{90-9} = 9$$
$$\sqrt{81} = 9$$
$$9 = 9$$

This is true, so $x = 9$ is a solution.

4.
$$\sqrt{8x-12} + 5x = 6x$$
$$\sqrt{8x-12} = x$$
$$8x - 12 = x^2$$
$$x^2 - 8x + 12 = 0$$
$$(x-6)(x-2) = 0$$

So $x = 6$ and $x = 2$ are our potential solutions.

Check them.

$$\sqrt{8(6)-12} + 5(6) = 6(6)$$
$$\sqrt{48-12} + 30 = 36$$
$$\sqrt{36} + 30 = 36$$
$$6 + 30 = 36$$
$$36 = 36$$

So $x = 6$ is a solution.

$$\sqrt{8(2)-12}+5(2)=6(2)$$
$$\sqrt{16-12}+10=12$$
$$\sqrt{4}+10=12$$
$$2+10=12$$
$$12=12$$

So $x = 2$ is also a solution.

5.
$$-4x-2=-5x+3+\sqrt{3x-11}$$
$$x-5=\sqrt{3x-11}$$
$$(x-5)^2=3x-11$$
$$x^2-10x+25=3x-11$$
$$x^2-13x+36=0$$
$$(x-9)(x-4)=0$$

So $x = 4$ and $x = 9$ are our potential solutions.

Check them.

$$-4(4)-2=-5(4)+3+\sqrt{3(4)-11}$$
$$-18=-20+3+\sqrt{1}$$
$$-18=-20+3+1$$
$$-18=-16$$

So $x = 4$ is not a solution.

$$-4(9)-2=-5(9)+3+\sqrt{3(9)-11}$$
$$-38=-45+3+\sqrt{16}$$
$$-38=-45+3+4$$
$$-38=-38$$

So $x = 9$ is a solution.

6.

$$14 = 34 - 5\sqrt[3]{2x}$$
$$5\sqrt[3]{2x} = 20$$
$$\sqrt[3]{2x} = 4$$
$$2x = 64$$
$$x = 32$$

Check it.

$$14 = 34 - 5\sqrt[3]{2(32)}$$
$$14 = 34 - 5\sqrt[3]{64}$$
$$14 = 34 - 5(4)$$
$$14 = 34 - 20$$
$$14 = 14$$

So $x = 32$ is a solution.

7.

$$\sqrt[3]{x^3 + x^2 + 13x + 42} = x$$
$$x^3 + x^2 + 13x + 42 = x^3$$
$$x^2 + 13x + 42 = 0$$
$$(x + 7)(x + 6) = 0$$

So $x = -7$ and $x = -6$ are our potential solutions.

Check them.

$$\sqrt[3]{(-7)^3 + (-7)^2 + 13(-7) + 42} = -7$$
$$\sqrt[3]{-343 + 49 - 91 + 42} = -7$$
$$\sqrt[3]{-343} = -7$$
$$-7 = -7$$

So $x = -7$ is a solution.

$$\sqrt[3]{(-6)^3 + (-6)^2 + 13(-6) + 42} = -6$$
$$\sqrt[3]{-216 + 36 - 78 + 42} = -6$$
$$\sqrt[3]{-216} = -6$$
$$-6 = -6$$

So $x = -6$ is also a solution.

8.

$$\sqrt[4]{x^4 + x^2 + 13x + 42} = x$$
$$x^4 + x^2 + 13x + 42 = x^4$$
$$x^2 + 13x + 42 = 0$$
$$(x+6)(x+7) = 0$$

So $x = -7$ and $x = -6$ are our potential solutions.

.Check them.

$$\sqrt[4]{(-7)^4 + (-7)^2 + 13(-7) + 42} = -7$$
$$\sqrt[4]{2401 + 49 - 91 + 42} = -7$$
$$\sqrt[4]{2401} = -7$$
$$7 = -7$$

This is not true, so $x = -7$ is not a solution. Similarly, $x = -6$ is not a solution. So there are no solutions to the original equation.

9.

$$\sqrt[5]{32x^5 + x^2 - 2x - 80} = 2x$$
$$32x^5 + x^2 - 2x - 80 = (2x)^5$$
$$32x^5 + x^2 - 2x - 80 = 32x^5$$
$$x^2 - 2x - 80 = 0$$
$$(x-10)(x+8) = 0$$

So $x = -8$ and $x = 10$ are our potential solutions.

Check them.

$$\sqrt[5]{32(-8)^5 + (-8)^2 - 2(-8) - 80} = 2(-8)$$
$$\sqrt[5]{32(-32768) + 64 + 16 - 80} = -16$$
$$\sqrt[5]{-1048576} = -16$$
$$-16 = -16$$

So $x = -8$ is a solution.

$$\sqrt[5]{32(10)^5 + 10^2 - 2(10) - 80} = 2(10)$$
$$\sqrt[5]{3200000 + 100 - 20 - 80} = 20$$
$$\sqrt[5]{3200000} = 20$$
$$20 = 20$$

So $x = 10$ is also a solution.

10. $\sqrt[6]{x^6 + x^3 + 2x^2 - x - 2} = x$

$x^6 + x^3 + 2x^2 - x - 2 = x^6$

$x^3 + 2x^2 - x - 2 = 0$

If this equation has any rational solutions, we know they must be 1, −1, 2, or −2. After checking these, we find that $x = -2$, $x = -1$, and $x = 1$ are all solutions of this simplified equation. We must then check that each is a solution of the original equation. It is true that $x = 1$ is indeed a solution, but $x = -2$ and $x = -1$ are not solutions. So the only solution of the original equation is $x = 1$.

Lesson 27

1. Since 4 is even, we must make sure that $x + 3 \geq 0$ when considering those values of x that are in the domain. Thus, the domain of this function is $[-3, \infty)$.

2. Because 5 is odd, we know that the domain of this function is the set of all real numbers.

3. Because 3 is odd, we know that the domain of this function is the set of all real numbers.

4. Because 7 is odd, we know that the domain of this function is the set of all real numbers.

5. Because 3 is odd, we know that the domain of this function is the set of all real numbers.

6. This is the graph of $x^{1/4}$ shifted to the left 3 units and then up 2 units. Moreover, the graph of $x^{1/4}$ has a similar shape to that of $x^{1/2}$, so the graph looks like the following.
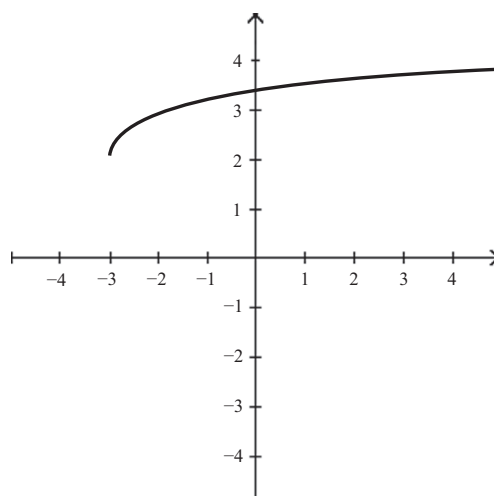
7. This is the same as the graph of $x^{1/5}$ except shifted down 1 unit. Moreover, the graph of $x^{1/5}$ has a similar shape to that of $x^{1/3}$, so the graph looks like the following.
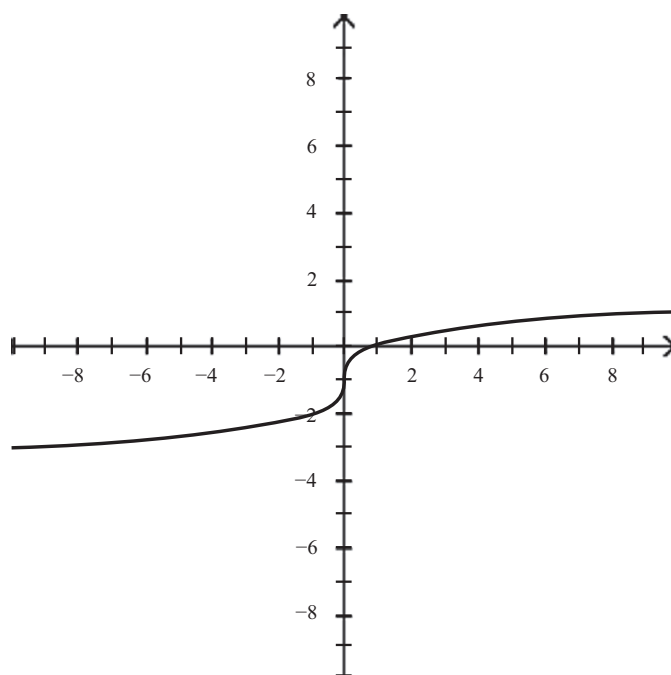
8. This is the same function as $(x-1)^{1/3} + 6$. So its graph is the graph of $x^{1/3}$ shifted 1 unit to the right and 6 units up.
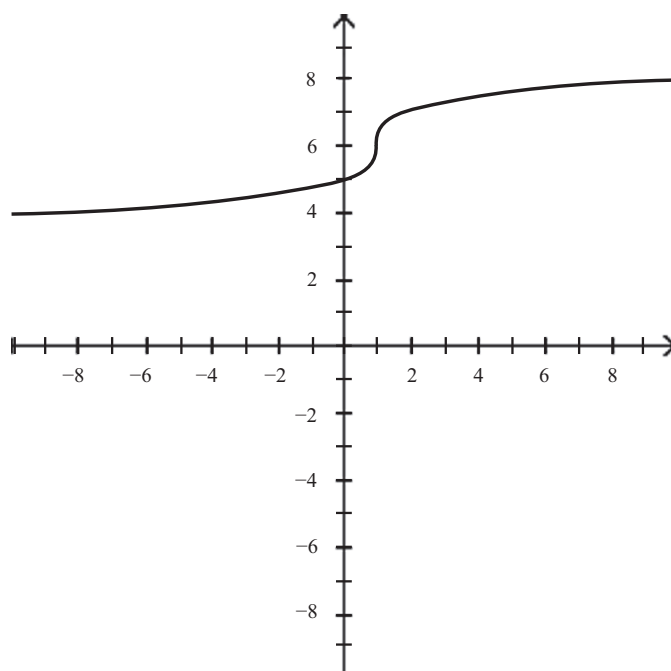
9. Since 7 is odd and 4 is even, this graph will have a similar structure to the graph of $x^{2/3}$. This graph looks like the following.
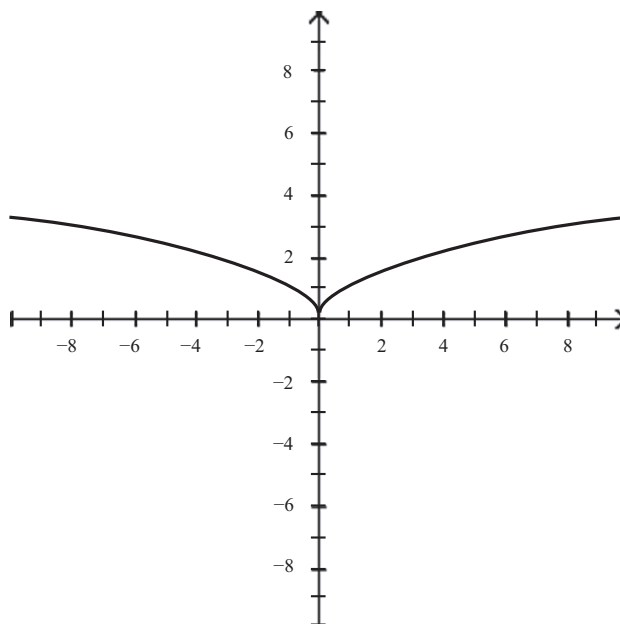
10. This graph can be built as follows: Start with the graph of $x^{1/3}$, shift it 2 units to the right, then flip it over the x-axis, and finally shift it up 4 units.
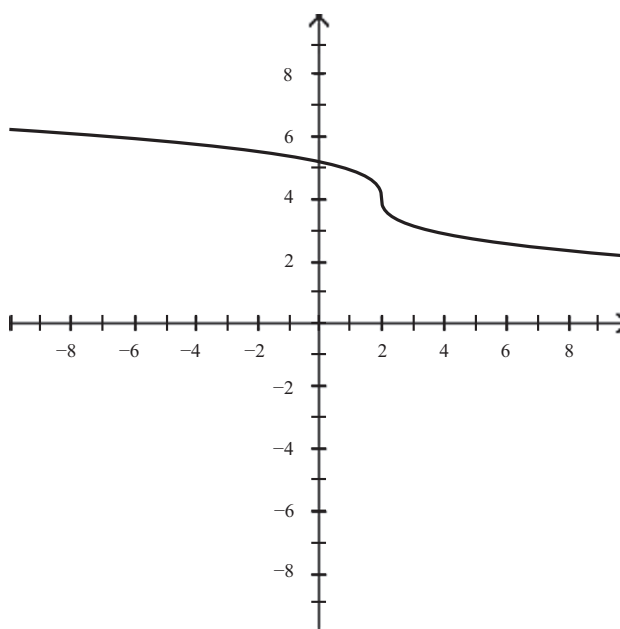
1. $f(x) = \dfrac{2x+1}{x-1}$

domain: all real numbers except $x = 1$

horizontal asymptote: $y = 2$

vertical asymptote: $x = 1$

2. $f(x) = 2 + \dfrac{7}{x+3}$

We begin by rewriting the function as follows.

$$f(x) = 2 + \frac{7}{x+3}$$
$$f(x) = \frac{2(x+3)}{x+3} + \frac{7}{x+3}$$
$$f(x) = \frac{2x+13}{x+3}$$

domain: all real numbers except $x = -3$

horizontal asymptote: $y = 2$

vertical asymptote: $x = -3$

3. $f(x) = \dfrac{-6}{x^2 - 4x}$

domain: all real numbers except $x = 0$ and $x = 4$

(You can see this once you've factored the denominator.)

horizontal asymptote: $y = 0$

(since the degree of the numerator is less than the degree of the denominator)

vertical asymptotes: $x = 0$, $x = 4$

4. $f(x) = \dfrac{4x^2 - 16x}{x^2 - 2x - 3}$

Notice that the function factors as follows.

$$f(x) = \frac{4x^2 - 16x}{x^2 - 2x - 3}$$
$$f(x) = \frac{4x(x - 4)}{(x - 3)(x + 1)}$$

domain: all real numbers except $x = -1$ and $x = 3$

horizontal asymptote: $y = 4$

vertical asymptotes: $x = -1$ and $x = 3$

5. $f(x) = \dfrac{5x^2 + 25x + 30}{x^2 + 3x + 2}$

Notice that the function factors as follows.

$$f(x) = \frac{5x^2 + 25x + 30}{x^2 + 3x + 2}$$
$$f(x) = \frac{5(x^2 + 5x + 6)}{(x + 2)(x + 1)}$$
$$f(x) = \frac{5(x + 2)(x + 3)}{(x + 2)(x + 1)}$$
$$f(x) = \frac{5(x + 3)}{x + 1}$$

domain: all real numbers except $x = -2$ and $x = -1$.

(Remember: Even though the factors of $x + 2$ were canceled in the most simplified version of the function, we still cannot plug $x = -2$ into the original function. So we must remove $x = -2$ from the domain of this function.)

horizontal asymptote: $y = 5$

vertical asymptote(s): $x = -1$

Note that there will not be a vertical asymptote at $x = -2$, but there will be a hole in the graph at $(-2, -5)$.

6. $f(x) = \dfrac{2x+1}{x-1}$

Using the information from the earlier problem, as well as the *x*-intercept(s) and the *y*-intercept, we have the following.
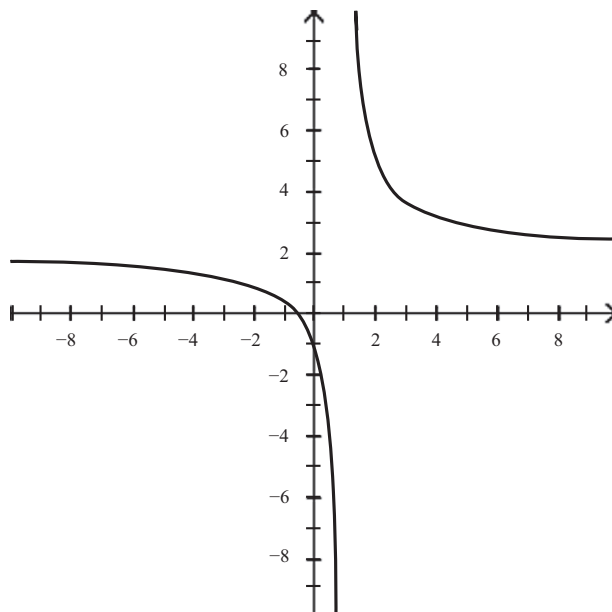
7. $f(x) = 2 + \dfrac{7}{x+3}$

Using the information from the earlier problem, as well as the *x*-intercept(s) and the *y*-intercept, we have the following.
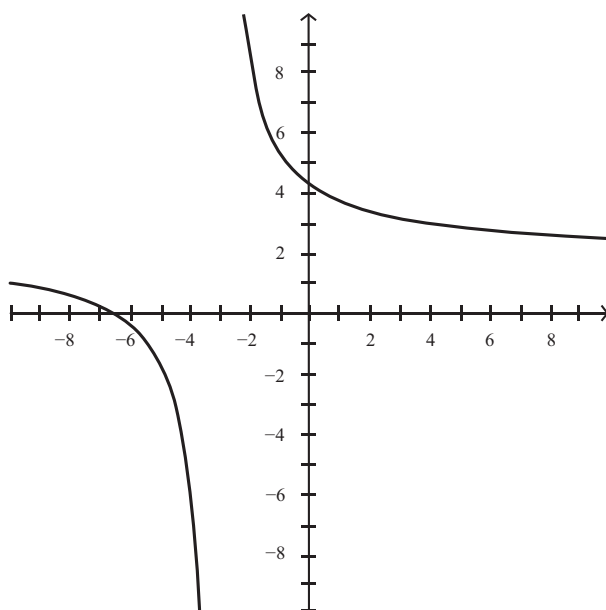
8. $f(x) = \dfrac{-6}{x^2 - 4x}$

Using the information from the earlier problem, as well as the x-intercept(s) and the y-intercept, we have the following.
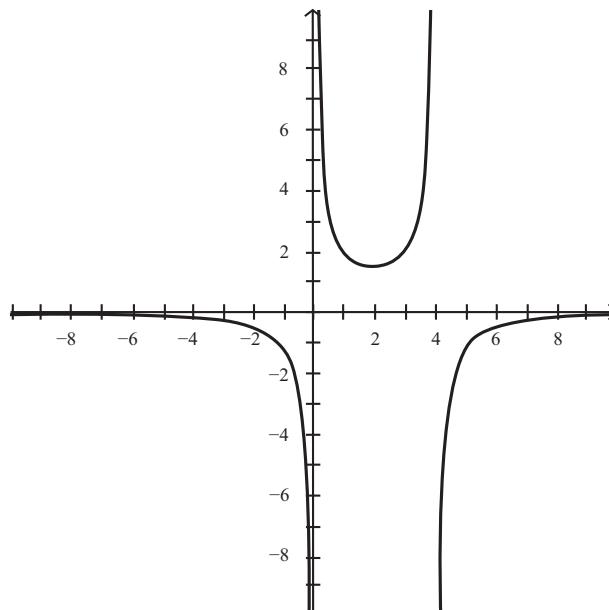
9. $f(x) = \dfrac{4x^2 - 16x}{x^2 - 2x - 3}$

Using the information from the earlier problem, as well as the x-intercept(s) and the y-intercept, we have the following.
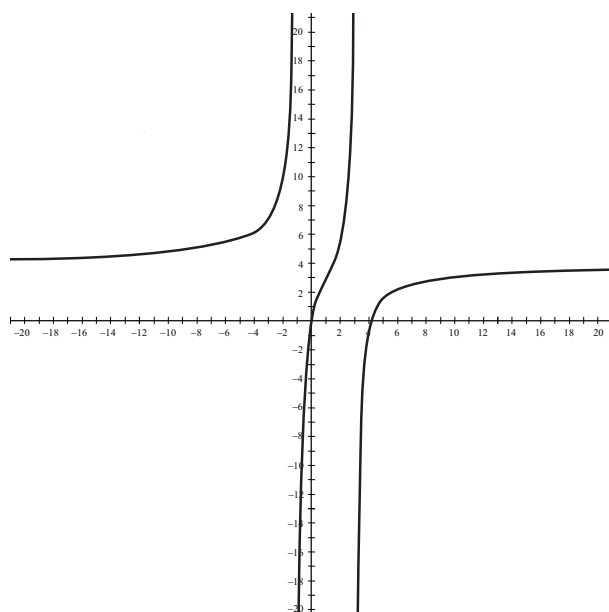
Are you wondering why the middle line of the graph is allowed to cross the horizontal asymptote at $y = 4$? There is a common misconception about horizontal asymptotes: Their noncrossing nature is only true at the ends; in the middle of the graph, all sorts of crossings can occur!

10. $f(x) = \dfrac{5x^2 + 25x + 30}{x^2 + 3x + 2}$

Using the information from the earlier problem, as well as the x-intercept(s) and the y-intercept, we have the following.

1.
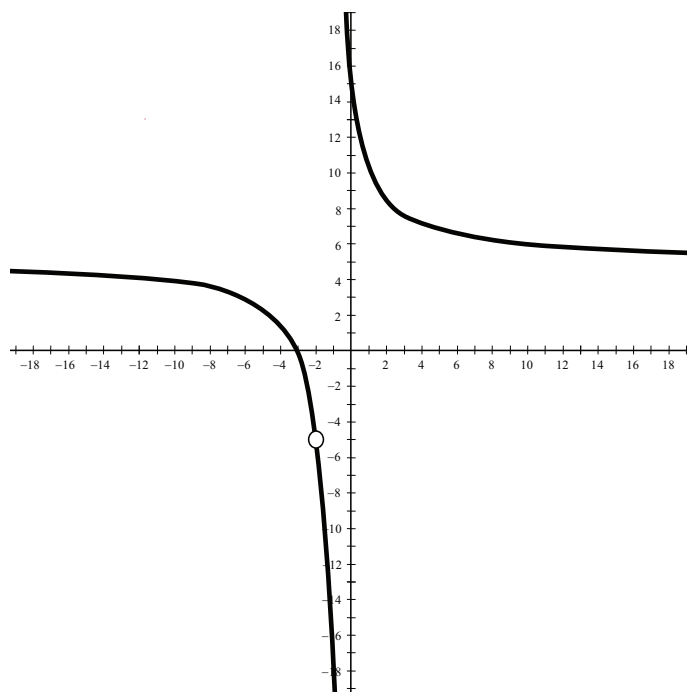
$$\frac{2}{x^2-9x+14}+\frac{5}{x^2-12x+20}$$

$$=\frac{2}{(x-2)(x-7)}+\frac{5}{(x-2)(x-10)}$$

$$=\frac{2(x-10)}{(x-2)(x-7)(x-10)}+\frac{5(x-7)}{(x-2)(x-7)(x-10)}$$

$$=\frac{2(x-10)+5(x-7)}{(x-2)(x-7)(x-10)}$$

$$=\frac{2x-20+5x-35}{(x-2)(x-7)(x-10)}$$

$$=\frac{7x-55}{(x-2)(x-7)(x-10)}$$

2.

$$7+\frac{x-1}{x-3}$$

$$=\frac{7(x-3)}{x-3}+\frac{x-1}{x-3}$$

$$=\frac{7x-21+x-1}{x-3}$$

$$=\frac{8x-22}{x-3}$$

$$=\frac{2(4x-11)}{x-3}$$

3.

$$\frac{x+5}{x-5}-\frac{x+2}{x-2}$$

$$=\frac{(x+5)(x-2)}{(x-5)(x-2)}-\frac{(x+2)(x-5)}{(x-5)(x-2)}$$

$$=\frac{x^2+3x-10}{(x-5)(x-2)}-\frac{x^2-3x-10}{(x-5)(x-2)}$$

$$=\frac{x^2+3x-10-x^2+3x+10}{(x-5)(x-2)}$$

$$=\frac{6x}{(x-5)(x-2)}$$

4.

$$\frac{3x}{x^2-16}-\frac{2x+1}{x^2+6x+8}$$

$$=\frac{3x}{(x-4)(x+4)}-\frac{2x+1}{(x+4)(x+2)}$$

$$=\frac{3x(x+2)}{(x-4)(x+4)(x+2)}-\frac{(2x+1)(x-4)}{(x-4)(x+4)(x+2)}$$

$$=\frac{3x^2+6x}{(x-4)(x+4)(x+2)}-\frac{2x^2-7x-4}{(x-4)(x+4)(x+2)}$$

$$=\frac{3x^2+6x-2x^2+7x+4}{(x-4)(x+4)(x+2)}$$

$$=\frac{x^2+13x+4}{(x-4)(x+4)(x+2)}$$

5.

$$\frac{x^2-16}{x+9}\cdot\frac{x^2-x-90}{x^2+19x+60}$$

$$=\frac{(x-4)(x+4)}{x+9}\cdot\frac{(x-10)(x+9)}{(x+15)(x+4)}$$

$$=\frac{(x-4)(x-10)}{(x+15)}$$

6.

$$\frac{x^3-13x^2+42x}{x^2-49}\cdot\left(x^2+8x+7\right)$$

$$=\frac{x^3-13x^2+42x}{x^2-49}\cdot\left(\frac{x^2+8x+7}{1}\right)$$

$$=\frac{x(x-7)(x-6)}{(x-7)(x+7)}\cdot\left(\frac{(x+7)(x+1)}{1}\right)$$

$$=x(x-6)(x+1)$$

7.

$$\frac{x+3}{x+2}\div\frac{x^2+2x-3}{x^2-2x+1}$$

$$=\frac{x+3}{x+2}\cdot\frac{(x-1)^2}{(x+3)(x-1)}$$

$$=\frac{x-1}{x+2}$$

8.

$$\frac{x^2+x-56}{x^2-2x-80} \div \frac{1}{x+8}$$

$$= \frac{x^2+x-56}{x^2-2x-80} \cdot \frac{x+8}{1}$$

$$= \frac{(x+8)(x-7)}{(x-10)(x+8)} \cdot \frac{x+8}{1}$$

$$= \frac{(x+8)(x-7)}{x-10}$$

9.

$$f(x) = \frac{x}{x+1}, g(x) = \frac{x}{x-1}$$

$$f(g(x)) = f\left(\frac{x}{x-1}\right)$$

$$= \frac{\dfrac{x}{x-1}}{\dfrac{x}{x-1}+1}$$

$$= \frac{\dfrac{x}{x-1}}{\dfrac{x}{x-1}+\dfrac{x-1}{x-1}}$$

$$= \frac{\dfrac{x}{x-1}}{\dfrac{2x-1}{x-1}}$$

$$= \frac{x}{2x-1}$$

10.

$$f(x) = \frac{2}{x}, g(x) = \frac{2x}{x^2-1}$$

$$f(g(x)) = f\left(\frac{2x}{x^2-1}\right)$$

$$= \frac{2}{\dfrac{2x}{x^2-1}}$$

$$= \frac{2}{1} \cdot \frac{x^2-1}{2x}$$

$$= \frac{x^2-1}{x}$$

1. Since $\dfrac{5x+7}{x^2+2x-3} = \dfrac{5x+7}{(x+3)(x-1)}$, the decomposition starts as $\dfrac{A}{x+3} + \dfrac{B}{x-1}$.

2. $\dfrac{A}{x+2} + \dfrac{B}{(x+2)^2}$

3. $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$

4. $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-3}$

5. Since $\dfrac{7x^2-16x+36}{x^4-16} = \dfrac{7x^2-16x+36}{(x-2)(x+2)(x^2+4)}$, the decomposition starts as $\dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{Cx+D}{x^2+4}$.

6. $\dfrac{5x+7}{(x+3)(x-1)}$

$= \dfrac{A}{x+3} + \dfrac{B}{x-1}$

$= \dfrac{A(x-1)}{(x+3)(x-1)} + \dfrac{B(x+3)}{(x+3)(x-1)}$

$= \dfrac{Ax-A+Bx+3B}{(x+3)(x-1)}$

So we know the following.

$$A+B = 5$$
$$-A+3B = 7$$

Solving this system gives us $A = 2$, $B = 3$. This means our partial fraction decomposition is $\dfrac{2}{x+3} + \dfrac{3}{x-1}$.

7.

$$\frac{3x+8}{(x+2)^2}$$

$$=\frac{A}{x+2}+\frac{B}{(x+2)^2}$$

$$=\frac{A(x+2)}{(x+2)^2}+\frac{B}{(x+2)^2}$$

$$=\frac{Ax+2A+B}{(x+2)^2}$$

So $A = 3$ and $2A + B = 8$. This means $A = 3$ and $B = 2$. So our partial fraction decomposition is $\dfrac{3}{x+2}+\dfrac{2}{(x+2)^2}$.

8.

$$\frac{2x}{(x+1)(x^2+1)}$$

$$=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$

$$=\frac{A(x^2+1)}{(x+1)(x^2+1)}+\frac{(Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$$=\frac{Ax^2+A+Bx^2+Bx+Cx+C}{(x+1)(x^2+1)}$$

So we know the following.

$$A+B=0$$
$$B+C=2$$
$$A+C=0$$

Solving this system tells us that $A = -1$, $B = 1$, and $C = 1$. So our partial fraction decomposition is $=\dfrac{-1}{x+1}+\dfrac{x+1}{x^2+1}$.

9.

$$\frac{36}{x^2(x-3)}$$

$$=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-3}$$

$$=\frac{Ax(x-3)}{x^2(x-3)}+\frac{B(x-3)}{x^2(x-3)}+\frac{Cx^2}{x^2(x-3)}$$

$$=\frac{Ax^2-3Ax+Bx-3B+Cx^2}{x^2(x-3)}$$

So we know the following.

$$A + C = 0$$
$$B - 3A = 0$$
$$-3B = 36$$

So we immediately see that $B = -12$. Then we know that $A = -4$, which means that $C = 4$. So our partial fraction

decomposition is $\dfrac{-4}{x} + \dfrac{-12}{x^2} + \dfrac{4}{x-3}$.

10. $\dfrac{7x^2 - 16x + 36}{x^4 - 16}$

$$= \dfrac{7x^2 - 16x + 36}{(x-2)(x+2)(x^2+4)}$$

$$= \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{Cx+D}{x^2+4}$$

$$= \dfrac{A(x+2)(x^2+4)}{(x-2)(x+2)(x^2+4)} + \dfrac{B(x-2)(x^2+4)}{(x-2)(x+2)(x^2+4)} + \dfrac{(Cx+D)(x-2)(x+2)}{(x-2)(x+2)(x^2+4)}$$

$$= \dfrac{Ax^3 + 4Ax + 2Ax^2 + 8A}{(x-2)(x+2)(x^2+4)} + \dfrac{Bx^3 + 4Bx - 2Bx^2 - 8B}{(x-2)(x+2)(x^2+4)} + \dfrac{Cx^3 - 4Cx + Dx^2 - 4D}{(x-2)(x+2)(x^2+4)}$$

$$= \dfrac{Ax^3 + 4Ax + 2Ax^2 + 8A + Bx^3 + 4Bx - 2Bx^2 - 8B + Cx^3 - 4Cx + Dx^2 - 4D}{(x-2)(x+2)(x^2+4)}$$

So we know the following.

$$A + B + C = 0$$
$$2A - 2B + D = 7$$
$$4A + 4B - 4C = -16$$
$$8A - 8B - 4D = 36$$

Solving this system gives us $A = 1, B = -3, C = 2$, and $D = -1$. So the partial fraction decomposition is

$\dfrac{1}{x-2} + \dfrac{-3}{x+2} + \dfrac{2x-1}{x^2+4}$.

1.
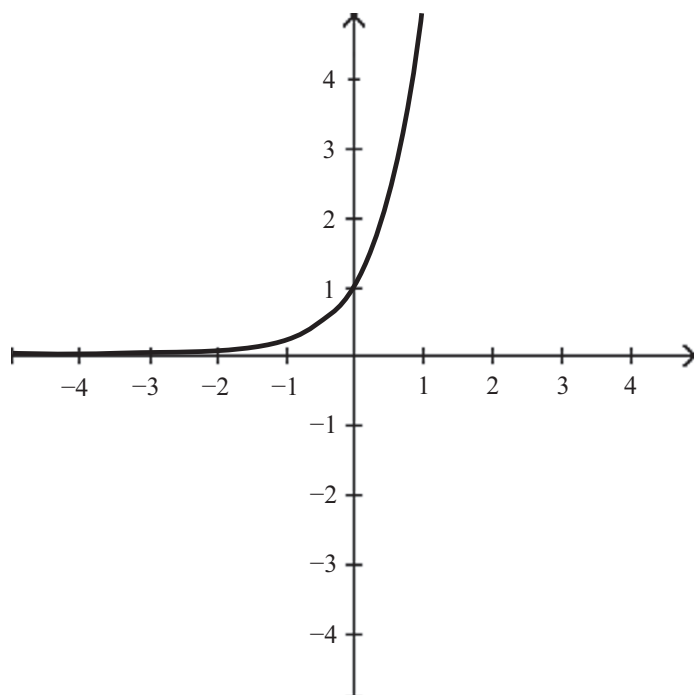
2.
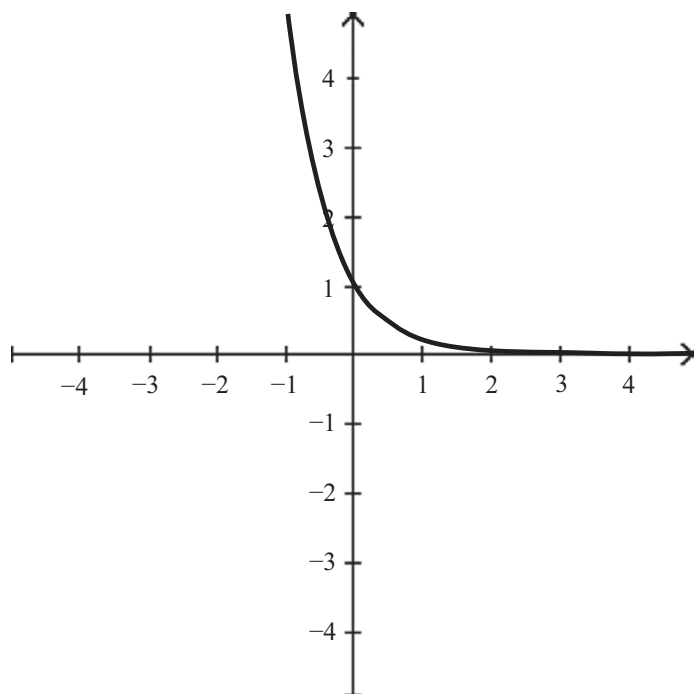
3.   This is the same as the graph of 2^x just shifted 6 units to the left. Note that the point $(-6, 1)$ is on the graph.
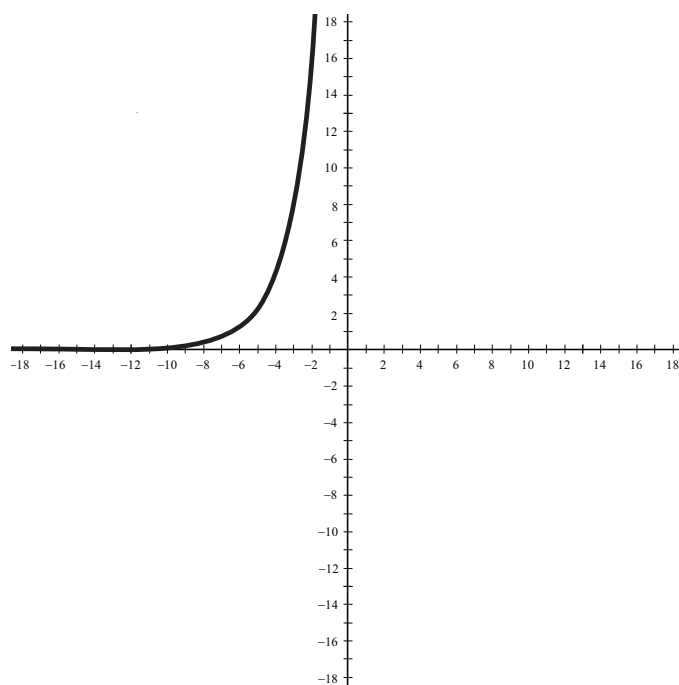
4.   This is the same as the graph of 4^x just shifted 2 units to the right and 3 units up.
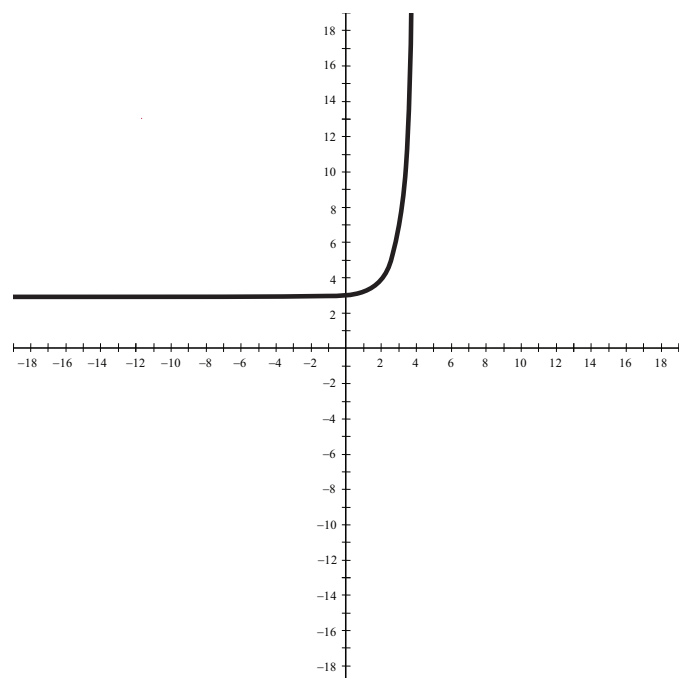
5. This is the same as the graph of $\left(\dfrac{1}{4}\right)^x$ just shifted 3 units down.
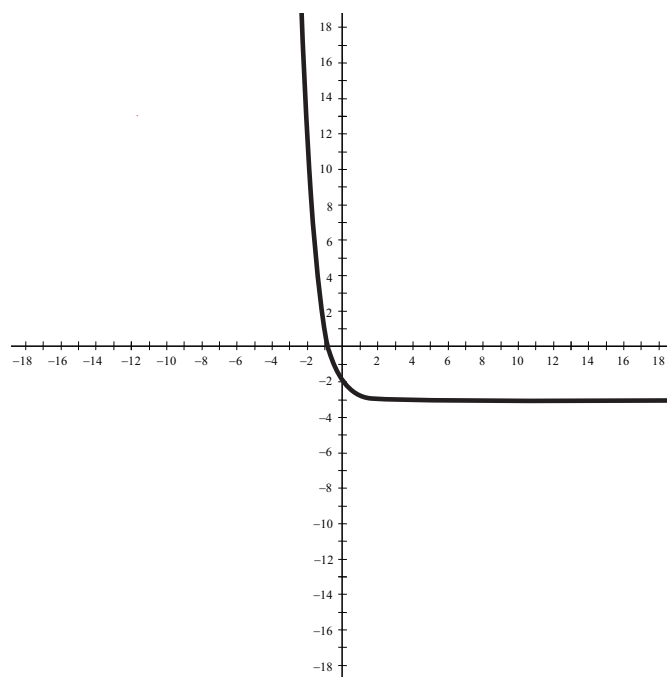
6. This is the same as the graph of 7^x, flipped over the x-axis, then shifted 2 units up.
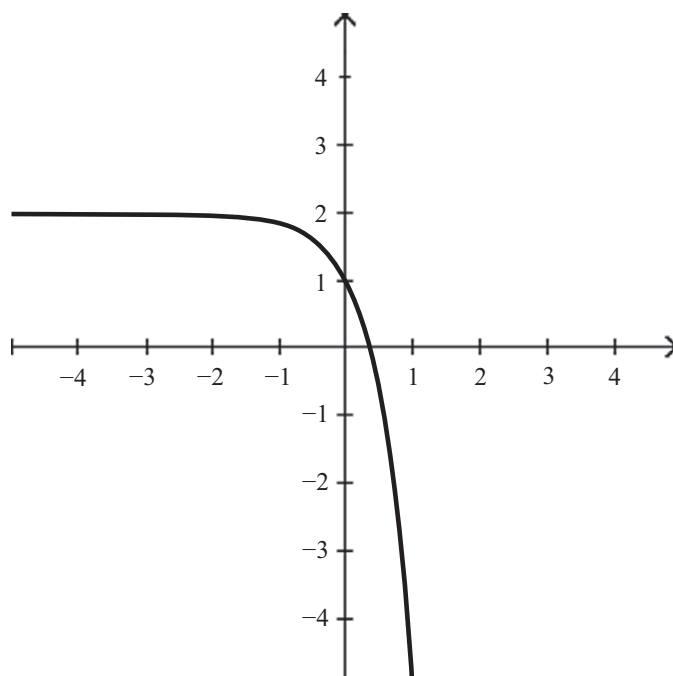
7. Plugging the information from the points into the equation $y = ab^x$ gives us the following.

$$\frac{5}{8} = ab$$
$$\frac{125}{8} = ab^3$$

Dividing the second equation by the first and simplifying gives the following.

$$b^2 = \frac{\frac{125}{8}}{\frac{5}{8}} = \frac{125}{5} = 25$$

So $b = 5$. Plugging this back into one of the original equations, we have the following.

$$\frac{5}{8} = a(5)$$
$$a = \frac{1}{8}$$

So our function is $y = \frac{1}{8} \cdot 5^x$.

8. Plugging the information from the points into the equation $y = ab^x$ gives us the following.

$$7 = ab^0$$
$$28 = ab^{-2}$$

The first equation immediately tells us that $a = 7$. So the function is $y = 7 \cdot b^x$. Also, the second equation in the original system is now $28 = 7b^{-2}$. So $b^{-2} = 4$, which means that $b^{-1} = 2$ or $b = \frac{1}{2}$. So our function is $y = 7 \cdot \left(\frac{1}{2}\right)^x$.

9.   Plugging the information from the points into the equation $y = ab^x$ gives us the following.

$$-6 = ab$$
$$-48 = ab^4$$

Dividing the second equation by the first and simplifying gives the following.

$$b^3 = \frac{-48}{-6} = 8$$

So $b = 2$. Plugging this back into one of the original equations, we have the following.

$$-6 = a(2)$$
$$a = -3$$

So our function is $y = -3 \cdot 2^x$.

10.   Plugging the information from the points into the equation $y = ab^x$ gives us the following.

$$\frac{1}{36} = ab^2$$
$$\frac{3}{4} = ab^{-1}$$

Dividing the first equation by the second and simplifying gives the following.

$$b^3 = \frac{\dfrac{1}{36}}{\dfrac{3}{4}} = \frac{1}{36} \cdot \frac{4}{3} = \frac{1}{27}$$

So $b = \dfrac{1}{3}$. Plugging this back into one of the original equations, we have the following.

$$\frac{1}{36} = a\left(\frac{1}{3}\right)^2 = a\left(\frac{1}{9}\right)$$
$$a = \frac{9}{36} = \frac{1}{4}$$

So our function is $y = \frac{1}{4} \cdot \left(\frac{1}{3}\right)^x$.

1. $\log_5 625 = \log_5 (5^4) = 4\log_5 5 = 4\cdot 1 = 4$

2. $\log_8 4 = \log_8 2^2 = \log_8 (8^{1/3})^2 = \log_8 (8^{2/3}) = \dfrac{2}{3}\log_8 8 = \dfrac{2}{3}\cdot 1 = \dfrac{2}{3}$

3. $\log_3 \left(\dfrac{1}{81}\right) = \log_3 \left(\dfrac{1}{3^4}\right) = \log_3 1 - \log_3 3^4 = 0 - 4\log_3 3 = -4\cdot 1 = -4$

4. $\log_7 \left(\dfrac{x^2 y^3}{t^5 z}\right)$

    $= \log_7 x^2 + \log_7 y^3 - \log_7 t^5 - \log_7 z$

    $= 2\log_7 x + 3\log_7 y - 5\log_7 t - \log_7 z$

5. $\log_2 \left(\dfrac{8a^5}{10b^3}\right)$

    $= \log_2 8 + 5\log_2 a - \log_2 10 - 3\log_2 b$

    $= 3 + 5\log_2 a - (\log_2 2 + \log_2 5) - 3\log_2 b$

    $= 3 + 5\log_2 a - \log_2 2 - \log_2 5 - 3\log_2 b$

    $= 3 + 5\log_2 a - 1 - \log_2 5 - 3\log_2 b$

    $= 2 + 5\log_2 a - \log_2 5 - 3\log_2 b$

6. $-3\log_4 (x) + 5\log_4 (y) - \log_4 (z)$

    $= \log_4 (x^{-3}) + \log_4 (y^5) + \log_4 (z^{-1})$

    $= \log_4 (\dfrac{1}{x^3}) + \log_4 (y^5) + \log_4 (\dfrac{1}{z})$

    $= \log_4 \left(\dfrac{y^5}{x^3 z}\right)$

7. $\log_5 x \cdot \log_5 y$

No property of logarithms will allow this to be combined into one logarithm.

8. $y = \log_2(x-1)$

This is the same as the graph of $\log_2 x$, except it is shifted 1 unit to the right. So the x-intercept of this graph is (2, 0) rather than (1, 0).
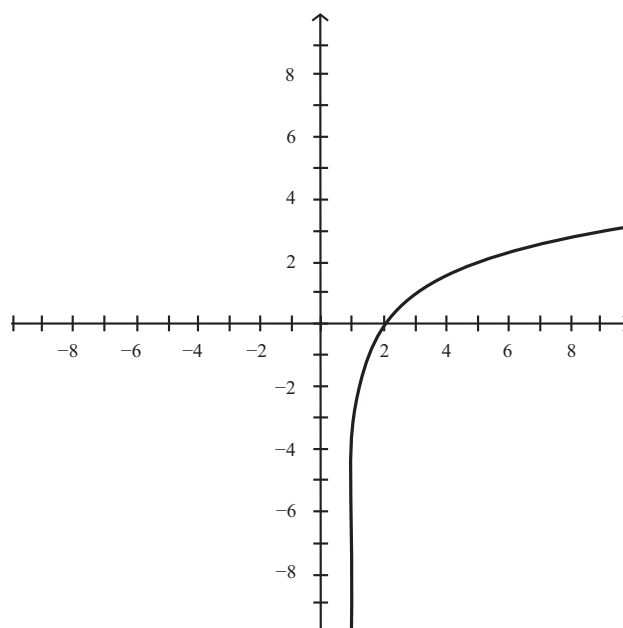
9. $y = \log_5(x+3)+2$

This is the same as the graph of $\log_5 x$, except it is shifted to the left 3 units and up 2 units.
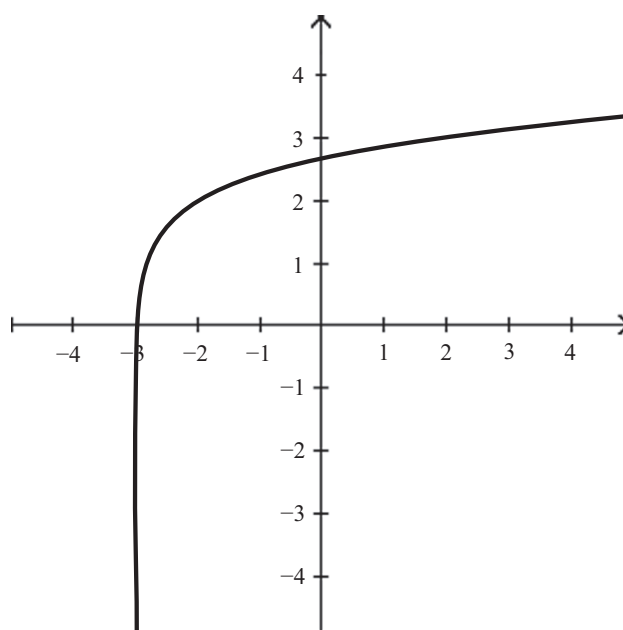
10. $y = -\log_3(x)$

This is the same as the graph of $\log_3 x$, except it is flipped about the *x*-axis.
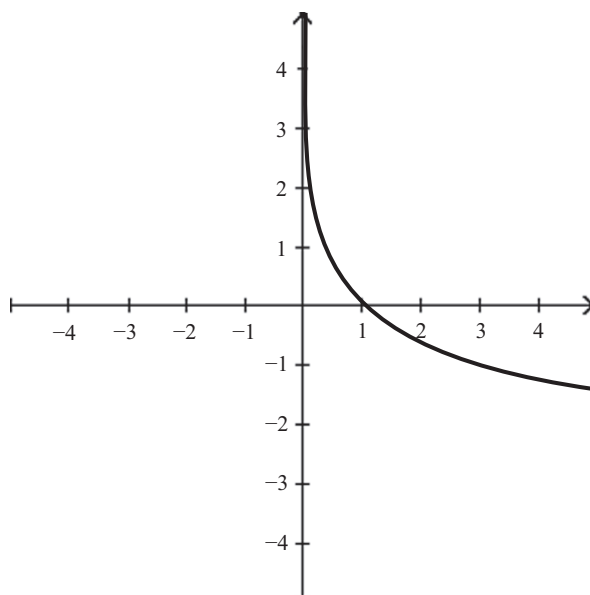
Lesson 33

1.
$$11^{x-8} - 5 = 42$$
$$11^{x-8} = 47$$
$$\log_{11}(11^{x-8}) = \log_{11} 47$$
$$x - 8 = \log_{11} 47$$
$$x = 8 + \log_{11} 47$$

If we wish to use a calculator to estimate this value, we can use the change of base formula to convert the logarithm to common logarithms.

$$x = 8 + \log_{11} 47 = 8 + \frac{\log 47}{\log 11} \approx 9.6056$$

2.
$$7 \cdot 6^{3x} = 42$$
$$6^{3x} = 6$$
$$3x = 1$$
$$x = \frac{1}{3}$$

3.
$$10e^{2x-10} - 4 = 70$$
$$10e^{2x-10} = 74$$
$$e^{2x-10} = 7.4$$
$$\ln(e^{2x-10}) = \ln 7.4$$
$$2x - 10 = \ln 7.4$$
$$2x = 10 + \ln 7.4$$
$$x = \frac{10 + \ln 7.4}{2}$$
$$x \approx 6.00074$$

4.
$$5^{4x+1} = 100$$
$$\log_5(5^{4x+1}) = \log_5 100$$
$$4x + 1 = \log_5 100$$
$$4x = \log_5 100 - 1$$
$$x = \frac{\log_5 100 - 1}{4}$$
$$x = \frac{\dfrac{\log 100}{\log 5} - 1}{4}$$
$$x \approx 0.4653$$

5.
$$\log_3(x^2 + 8) - \log_3 4 = 3$$
$$\log_3\left(\frac{x^2 + 8}{4}\right) = 3$$
$$\frac{x^2 + 8}{4} = 3^3$$
$$x^2 + 8 = 108$$
$$x^2 = 100$$
$$x = \pm 10$$

6.
$$\ln(x + 7) + \ln(x + 3) = \ln 77$$
$$\ln((x + 7)(x + 3)) = \ln 77$$
$$(x + 7)(x + 3) = 77$$
$$x^2 + 10x + 21 = 77$$
$$x^2 + 10x - 56 = 0$$
$$(x + 14)(x - 4) = 0$$

So $x = -14$ and $x = 4$ are the possible solutions. But x cannot equal -14; that would give us the natural logarithm of a negative number when we plug it in. So the only solution is $x = 4$.

7.
$$-3\log_5(x+1) = -12$$
$$\log_5(x+1) = 4$$
$$x+1 = 5^4$$
$$x+1 = 625$$
$$x = 624$$

8.
$$\log(x^2 + 9) = \log(7x - 3)$$
$$x^2 + 9 = 7x - 3$$
$$x^2 - 7x + 12 = 0$$
$$(x-4)(x-3) = 0$$

So $x = 3$ and $x = 4$ are solutions. We can confirm both of them.

9.
$$A = Pe^{rt}$$
$$A = 2000e^{.065(5)}$$
$$A = \$2768.06$$

10.
$$A = Pe^{rt}$$
$$9110.59 = 5000e^{0.05t}$$
$$\frac{9110.59}{5000} = e^{0.05t}$$
$$\ln\left(\frac{9110.59}{5000}\right) = 0.05t$$
$$t = \frac{\ln\left(\frac{9110.59}{5000}\right)}{0.05}$$
$$t \approx 11.99999$$

So the money has been in the account for 12 years.

1. $\dfrac{12!}{10!} = \dfrac{12 \cdot 11 \cdot 10!}{10!} = 12 \cdot 11 = 132$

2. $\dfrac{16!}{13!} = \dfrac{16 \cdot 15 \cdot 14 \cdot 13!}{13!} = 16 \cdot 15 \cdot 14 = 3360$

3. $C(12,9) = \dfrac{12!}{9!(12-9)!} = \dfrac{12!}{9!3!} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9!}{9!3!} = \dfrac{12 \cdot 11 \cdot 10}{6} = 220$

4. $C(12,3) = \dfrac{12!}{3!(12-3)!} = \dfrac{12!}{9!3!} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9!}{9!3!} = \dfrac{12 \cdot 11 \cdot 10}{6} = 220$

5. $C(50,50) = \dfrac{50!}{50!(50-50)!} = \dfrac{50!}{50!0!} = \dfrac{1}{0!} = \dfrac{1}{1} = 1$

6. $C(25,24) = \dfrac{25!}{24!(25-24)!} = \dfrac{25!}{24!1!} = \dfrac{25 \cdot 24!}{24!1} = \dfrac{25}{1} = 25$

7. $C(9,6) = \dfrac{9!}{6!(9-6)!} = \dfrac{9!}{6!3!} = \dfrac{9 \cdot 8 \cdot 7 \cdot 6!}{6!3!} = \dfrac{9 \cdot 8 \cdot 7}{3!} = \dfrac{9 \cdot 8 \cdot 7}{6} = 84$

8. Since the values in the 6th row of Pascal's triangle are 1, 6, 15, 20, 15, 6, 1, we have $(x+1)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$.

9. Since the values in the 8th row of Pascal's triangle are 1, 8, 28, 56, 70, 56, 28, 8, 1, we have

$(x+1)^8 = x^8 + 8x^7 + 28x^6 + 56x^5 + 70x^4 + 56x^3 + 28x^2 + 8x + 1$.

10. In this problem, we must replace every instance of x in the binomial theorem with $2t$. Since the values in the 7th row of Pascal's triangle are 1, 7, 21, 35, 35, 21, 7, 1, we have

$$(2t+1)^7$$
$$= (2t)^7 + 7(2t)^6 + 21(2t)^5 + 35(2t)^4 + 35(2t)^3 + 21(2t)^2 + 7(2t) + 1$$
$$= 128t^7 + 448t^6 + 672t^5 + 560t^4 + 280t^3 + 84t^2 + 14t + 1.$$

Lesson 35

1. $P(10,10) = \dfrac{10!}{(10-10)!} = \dfrac{10!}{0!} = \dfrac{10!}{1} = 3,628,800$

2. $P(8,4) = \dfrac{8!}{(8-4)!} = \dfrac{8!}{4!} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$

3. $P(7,5) = \dfrac{7!}{(7-5)!} = \dfrac{7!}{2!} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$

4. $C(8,4) = \dfrac{8!}{4!(8-4)!} = \dfrac{8!}{4!4!} = \dfrac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70$

5. $C(7,5) = \dfrac{7!}{5!(7-5)!} = \dfrac{7!}{5!2!} = \dfrac{7 \cdot 6}{2 \cdot 1} = 21$

6. Order matters here, so the answer is $P(10,2) = 10 \cdot 9 = 90$.

7. Order matters here, so the answer is $P(12,12) = 12! = 479,001,600$.

8. Order clearly matters here, as "TEA" is different from "ATE" (for example). Since there are 8 distinct letters in TEACHING, the answer here is $P(8,3) = 8 \cdot 7 \cdot 6 = 336$.

9. This problem involves combinations along with the multiplication principle. First, note that there are $C(15,6)$ ways to choose the people who will order steak. This leaves us with 9 people who have yet to order. Then there are $C(9,5)$ ways to choose the people who will order chicken. This leaves us 4 people to choose to order fish, so there are $C(4,4)$ ways to pick this group. Now we use the multiplication principle to see that the final answer to the problem is $C(15,6) \cdot C(9,5) \cdot C(4,4)$ —or 630,630 total ways to complete the problem.

10. Nathan has $C(6,4)$ ways to choose his pants, $C(8,5)$ ways to choose his shirts, and $C(3,2)$ ways to choose his vests. By the multiplication principle, the total number of ways to choose all these clothes is $C(6,4) \cdot C(8,5) \cdot C(3,2)$, or 2520.

Lesson 36

1. The number of even numbers in the set is 50 (the numbers 2, 4, 6, … , 10). Therefore, the probability is 50/100, or 1/2.

2. The desired numbers are 3, 6, 9, … , 99, and there are 33 such numbers in the set. So the probability is 33/100.

3. The desired numbers are 7, 17, 27, … , 97, and there are 10 such numbers in the set. So the probability is 10/100, or 1/10.

4. 30/100, or 3/10

5. 60/100, or 3/5

6. There are 2 such cards in the deck, so the probability is 2/52 or 1/26.

7. Since there are 26 red cards and 26 black cards in the deck, the probability is $\dfrac{C(26,3) \cdot C(26,2)}{C(52,5)}$, which is approximately 0.325.

8.   The number of ways of choosing the 2 kings out of the 4 kings is $C(4, 2)$. The number of ways of choosing the 2 queens out of the 4 queens is also $C(4, 2)$. The number of ways to choose the fifth card in the hand is $C(44, 1)$.

Therefore, the probability is $\dfrac{C(4,2) \cdot C(4,2) \cdot C(44,1)}{C(52,5)}$, which is approximately 0.00061.

9.   We choose one of the 13 kinds from which the 4 of a kind will be chosen. So there are $C(13, 1)$ choices for this kind. We then choose all 4 of those 4 cards. Now we must choose one additional card to complete the 5-card hand. There are $C(48, 1)$ ways to do this. So the final probability is

$\dfrac{C(13,1) \cdot C(48,1)}{C(52,5)}$, which is approximately 0.00024.

10.   We begin by choosing the suit from which we will choose the cards (since we want a flush). This can be done in $C(4, 1)$ ways. Next, we choose the smallest card in the hand (where the straight will start). There are 10 ways to do this (ace, 2, 3, 4, … , 10); no straight can start with a jack, queen, or king.

Therefore, the probability is $\dfrac{10 \cdot C(4,1)}{C(52,5)}$, which is approximately 0.000015.

Glossary

absolute value: The absolute value of a number x is typically denoted $|x|$ and is the distance from x to the origin on the number line.

asymptote: A straight line or curve that is the limiting value of the graph of a function. *See also* **horizontal asymptote** and **vertical asymptote**.

base (of a logarithm): The value b of the exponential function.

binomial: A polynomial containing exactly 2 terms.

Cartesian plane (or ***xy*-plane**): The plane or 2-dimensional space formed by 2 number lines, 1 horizontal and 1 vertical, so that they intersect at right angles.

center of the circle: The point from which all the points on a circle are equidistant (this is basically the point in the center of the circle).

circle: The set of all points that are a given distance from a special point (called the center of the circle).

combination: A selection of a set of items in which order does not matter.

complex number: A number consisting of a real and an imaginary part. The number can be written in the form $a + bi$, where a and b are real numbers, and i is the standard imaginary unit $i = \sqrt{-1}$.

composing: The process of replacing the variables in one function with a different function.

composition of 2 functions: For functions $f(x)$ and $g(x)$, this is the new function obtained by inserting $g(x)$ into the function $f(x)$. The new function is often denoted $f(g(x))$.

constant term: The term of a polynomial that has no variable.

cubic polynomial: A polynomial of degree 3.

decay factor: The specific term for the b value of the exponential function when $b < 1$.

degree of a polynomial: The largest power of the variable in the polynomial.

denominator: The bottom number in a fraction.

dependent system: A system of linear equations that has infinitely many solutions (because the 2 lines in the system are actually the same line).

difference of 2 squares: An equation that has one squared term subtracted from another squared term. There is no middle term in a difference of 2 squares.

directrix: A special line used in the definition of a parabola.

discriminant: The quantity $b^2 - 4ac$ for the quadratic equation $ax^2 + bx + c = 0$.

domain of a function: The set of input values of a function.

dominant term: The term in a polynomial that has the highest degree.

eccentricity of the ellipse: The length of the minor axis divided by the length of the major axis.

ellipse: The set of all points in the plane such that the sum of the distances from each of the points on the ellipse to the 2 foci is a constant amount.

exponential function: A function of the form $f(x) = ab^x$, where x is a real number, a is a nonzero constant, and b is a positive real number that is not equal to 1.

extraneous solution: A solution of a simplified version of an equation that is not a solution of the original equation.

factorial function: For a positive integer n, the function "n factorial" (which is denoted $n!$) is the product of all the integers from 1 to n.

focus: A special point used in the definition of parabolas and other conic sections.

function: A set of pairs of input values and output values in which each input value is assigned to exactly 1 output value.

growth factor: The specific term for the b value on the exponential function when $b > 1$.

horizontal asymptote: Horizontal asymptotes for a graph act like borders or guides for the graph when the x-values are large in either a positive or a negative direction.

hyperbola: The set of all points in the plane such that the differences of the distances from each of the points on the hyperbola to the 2 foci is a constant amount. The standard form of a hyperbola looks like $\begin{array}{l} y \le 5x+1 \\ y \ge 2x-3 \end{array}$ or $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$, where a and b are some real numbers. In the case of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, the vertices are at the points $(a,0)$ and $(-a,0)$ and the asymptotes are given by $y = \dfrac{b}{a}x$ and $y = -\dfrac{b}{a}x$. In the case of $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$, the vertices are at the points $(0,a)$ and $(0,-a)$ and the asymptotes are given by $y = \dfrac{a}{b}x$ and $y = -\dfrac{a}{b}x$.

identity: An equation that is true for every possible value of the variable.

imaginary number: A number whose square is a negative real number, often represented by the expression bi, where b is a real number and $y \ge 3$.

inconsistent system: A system of linear equations that has no solution.

independent system: A system of linear equations that has a unique solution.

leading coefficient of a polynomial: The coefficient (or number) in front of the term that contains the largest power of x in a polynomial.

linear polynomial: A polynomial of degree 1.

logarithm: If $y = b^x$, then the logarithm to the base b of a positive number y is denoted by $\log_b y$ and is defined by $\log_b y = x$.

major axis of the ellipse: The longer line segment in the axis of the ellipse.

minor axis of the ellipse: The shorter line segment in the axis of the ellipse.

monomial: A polynomial containing only 1 term.

n^{th} root: For any real numbers a and b, and any positive integer n, if $a^n = b$, then a is an nth root of b.

numerator: The top number in a fraction.

ordered pair: Two numbers inside parentheses, separated by a comma. The first number is the x-coordinate, or abscissa, and the second number is the y-coordinate, or ordinate, on a graph.

origin: The location where zero is placed on a real number line, and the point at which the axes intersect in the Cartesian plane.

parabola: The set of all points in the plane that are the same distance from the directrix as they are from the focus. A parabola is the graph of a quadratic function.

parallel: Two nonvertical lines with the same slope and different y-intercepts are said to be parallel to one another.

perfect square trinomial: A trinomial (3 terms) that can be written as the perfect square of a binomial (2 terms).

permutation: An arrangement of a set of items in a particular order.

perpendicular lines: Two lines that intersect at a right angle. If neither of the 2 lines is vertical, then we can say that the 2 lines are perpendicular if the slopes of the 2 lines are negative reciprocals of one another, or if the product of the 2 slopes equals −1.

point-slope form: The equation of a line that passes through the point $(x_1,\ y_1)$ and has slope m is $y - y_1 = m(x - x_1)$.

polynomial: An expression of the form $a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$, where each of the a's are real numbers (which could be positive, negative, or zero) and the powers on all the x's are positive integers.

probability: The number of ways an event can occur divided by the total number of possible outcomes. *See also* **theoretical probability**.

quadrants: The Cartesian plane is divided into four quadrants by the intersection of the x- and y-axes. They are called quadrants I, II, III, and IV. Quadrant I is the upper-right section, quadrant II is the upper-left section, quadrant III is the lower-left section, and quadrant IV is the lower-right section.

quadratic equation: An equation that includes a variable raised to the second power (i.e., squared). The typical form is $y = ax^2 + bx + c$.

quadratic function: A function of the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers.

quadratic inequality: An inequality that involves at least one term that is raised to the second power.

quadratic polynomial: A polynomial of degree 2.

quartic polynomial: A polynomial of degree 4.

quintic polynomial: A polynomial of degree 5.

range of the function: The set of output values of a function.

rational function: A ratio of 2 polynomial functions, or one polynomial divided by another polynomial.

sample space: The set of all possible outcomes for a particular activity or experiment.

slope: The rate of change of a line. Slope is defined as the vertical change of the line divided by the horizontal change of the line. This can also be thought of as "rise over run," where rise refers to the amount of vertical change and run is the amount of horizontal change; or it can be thought of as the change in y over the change in x. Slope is often represented as a fraction. When you have 2 points with coordinates (x_1, y_1) and (x_2, y_2) on the line, the slope is determined by $(y_2 - y_1)/(x_2 - x_1)$.

slope-intercept form: An equation of a line $y = mx + b$, where m is the slope of the line and b is the y-intercept of the line.

solution of a linear equation: A value of the variable that makes the equation true.

solution of a system: An ordered pair that makes all of the equations in a system of linear equations true.

standard form of a polynomial: The way of writing a polynomial such that the powers on the variables decrease as you read the polynomial from left to right.

strict inequality: An inequality that is only greater than or less than ($>$ or $<$), not greater than or equal to ($\geq$) or less than or equal to ($\leq$).

system of linear equations: A set of 2 or more linear equations.

theoretical probability: If a sample space has n equally likely outcomes and an event a occurs in m of these outcomes, then the theoretical probability of event a is denoted as $p(A)$ and determined as follows: $p(A) = \dfrac{m}{n}$.

trinomial: A polynomial containing exactly 3 terms.

variable: A symbol, usually a letter, that represents one or more numbers in an algebraic expression.

vertex of the parabola: The lowest point of a U-shaped parabola or the highest point of an upside-down U-shaped parabola.

vertical asymptote: If $x = c$ is a vertical asymptote of the graph of a function, then as the values of x get really close to c, the values of the function grow huge (going either to infinity or to negative infinity). Infinity is not actually shown on the graph.

vertical line test: A test used to determine whether a graph represents a function. If you can draw a vertical line that crosses through at least 2 points on a graph, then the graph does not represent a function.

vertices of the ellipse: The endpoints of the major axis of the ellipse.

x-axis: The horizontal number line, or horizontal axis.

x-coordinate (or **abscissa**): A number that shows where a point on a graph is located in relation to the x-axis.

x-intercepts: The points where the graph crosses the x-axis.

y-axis: The vertical number line, or vertical axis.

y-coordinate (or **ordinate**): A number that shows where a point on a graph is located in relation to the y-axis.

zeros of the equation: Values of x that are solutions of a quadratic equation; they are the x-intercepts of the graph $ax^2 + bx + c = 0$. They can be any number that when plugged into the quadratic equation, $ax^2 + bx + c = 0$, makes the equation true.

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